# Mergers and Collusion in All-Pay Auctions and Crowdsourcing Contests

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## ABSTRACT

We study the effects of bidder collaboration in all-pay auctions. We analyse both mergers, where the remaining players are aware of the agreement between the cooperating participants, and collusion, where the remaining players are unaware of this agreement. We examine two scenarios: the sum-profit model where the auctioneer obtains the sum of all submitted bids, and the max-profit model of crowdsourcing contests where the auctioneer can only use the best submissions and thus obtains only the winning bid. We show that while mergers do not change the expected utility of the participants, or the principal's utility in the sum-profit model, collusion transfers the utility from the non-colluders to the colluders. Surprisingly, we find that in some cases such collaboration can increase the social welfare. Moreover, mergers and, curiously, also collusion can even be beneficial to the auctioneer under certain conditions.

## Categories and Subject Descriptors

K.4 [**Computers and Society**]: Electronic Commerce; K.4.3 [**Computers and Society**]: Organizational Impacts—*Computer - supported collaborative work*

## Keywords

Collusion, Mergers, All-Pay Auction, Crowdsourcing

## 1. INTRODUCTION

Auctions are a key research area at the intersection of game theory, economics and computer science, which have recently received great attention in multi-agent systems as a powerful tool for task and resource allocation. In addition to explicit auctions such as those run in auction-houses or on the web, certain multi-player interactions can be modelled as "implicit" auctions. For example, firms competing in a race to issue a patent can be viewed as participating in a "latent" auction—the firm that invests the most in research effort is likely to issue the patent first, and secure itself the market [13, 8]. Similarly, workers employed by a firm may compete for a performance bonus or for titles such as "employee of the month" [11]. In the above examples, the player who exerts the most effort is likely to win the prize. However, the effort expended is itself costly to all players. Such scenarios therefore fall under the broad category of "all-pay auctions" [14, 23, 17], where *every* bidder pays his bid, but only the highest bid gets the reward [27, 10].

Recently, various firms employed a similar theme in so-called *crowdsourcing contests*. For instance, the video service Netflix issued a $1,000,000 prize in a contest to improve its film recommender system [12]. In this challenge, many movie recommendation algorithms were submitted, and the winning algorithm that offered the best performance under Netflix's criteria was chosen to receive the prize and replace the old algorithm. Another example is that of programming contests, such as those organised by Topcoder[1] or CodeChef.[2] In such contests, participants get to read a specification describing a problem to solve, design algorithms to best solve this problem, and submit their code to the principal running the contest. The winner is selected according to commonly known criteria, and only he wins a reward.

Crucially, in such scenarios the participants exert costly efforts in preparing their submissions, but most of that "goes to waste"—only the best submission is used by the principal, and only the author of the best submission is compensated for his work, in the form of a prize. Since the effort comes at a cost to the participant but increases his chance of winning, making a good decision regarding how much effort to exert is critical. Furthermore, in certain cases, the participants may find it worthwhile to pool their efforts and make a joint submission. Note that making several submissions would not make sense for the cooperating players, as only the highest quality one of these could possibly win the competition, while the others are guaranteed to be wasted effort.

One possible scenario of such collaboration is *mergers*, where the collaborating players declare that they are working together and preparing a single submission. In this case, the other participants can modify their behaviour accordingly. An alternative scenario is that of *collusion*, when the cooperating players keep their collaboration secret. If the

[1]www.topcoder.com
[2]www.codechef.com

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other participants are unaware of this collusion, we expect they would not deviate from an equilibrium behaviour, at least in the short run. In this case, the colluders can optimise the effort level they exert to maximise their utility. An important question is then, how would such collaboration affect the outcomes of the auction?

Against this background, we study the effects of mergers and collusion on all-pay auctions. Surprisingly, while these auctions were thoroughly investigated with respect to Nash equilibria [27, 10, 14, 17] and, on the other hand, bidder collaboration was shown to have a dramatic effect on the bidders' and the auctioneers' profits in many other settings [19], the impact of such collaboration in the context of all-pay auctions has been left relatively unexplored.

We distinguish between two auctioneer utility models:

1. *Sum-profit*—the auctioneer obtains bids from all participants, so his utility is given by the sum of the bids. One such setting is the "employee of the month" scenario, where the employer enjoys the efforts of all employees but only rewards the hardest-working one.

2. *Max-profit*—the auctioneer only uses the best submission, so his utility is determined by the maximal bid. The crowdsourcing contest falls under this category: all bidders put in effort in order to win, but the auctioneer only enjoys the effort of the best candidate.

## 1.1 Our Contribution

We first investigate the impact of mergers and collusion on the players' bidding strategies (representing the effort levels exerted). We then analyse the expected utilities of the participants and the principal running the auction. Further, we examine the change in variance of the above parameters due to mergers and collusion, which captures the risks for the bidders and the auctioneer.

Specifically, we show that the bidders' expected profits, or that of the auctioneer in the sum-profit model, do not change under mergers. However, mergers do increase the principal's utility in the max-profit model (although not as much as collusion, when the number of bidders is large). Collusion, in contrast, has a much bigger impact on player interaction. While in the case with no collaboration the participants exhaust all their surplus and get zero utility, if they collude, their expected profit is always positive. If there are only a few colluders, the outsiders' utilities may become negative, but if the number of colluders is large, even the non-colluders gain and obtain a positive utility.

Surprisingly, even the individual utility *per colluder* increases as their numbers grow, thus incentivising the bidders to join against the auctioneer. However, it turns out that under certain conditions the principal may even gain from collusion, both under the sum-profit and the max-profit models. Thus, as opposed to most auction settings where collusion limits competition and harms the auctioneer or the non-colluding bidders, in certain all-pay auctions all parties can actually gain from collusion.

Finally, we show how the effect of bidder collaboration on social welfare depends on the model of the principal's utility. In the sum-profit model, social welfare does not change under mergers or collusion. In the max-profit model, mergers increase the social welfare (as the principal's utility increases and bidder utility remains the same), but collusion may decrease social welfare, since the overall losses by non-colluders may be larger than the increase in the principal's utility.

The paper is organised as follows. In Section 2, we formally define an all-pay auction setting and characterise its symmetric equilibrium, which is required for our further analysis. The sections that follow then contain the main results regarding player collaboration in all-pay auctions: in Section 3 we discuss mergers, and we examine the impact of collusion in Section 4. In Section 5 we consider scenarios where some outsiders become aware of collusion and/or several colluding coalitions may form. Section 6 introduces the related literature, and Section 7 concludes.

# 2. MODEL AND PRELIMINARIES

We consider an all-pay auction with a single auctioned item that is commonly valued by all the participants. Our setting is a symmetric restricted case of the all-pay auction studied in [10], where players' values for the item may be different.

Formally, we assume that each of the $n$ bidders issues a bid of $b_i$, $i = 1, \ldots, n$, and all bidders value the item at 1. The highest bidders win the item and divide it among themselves, while the rest lose their bid. Thus, bidder $i$'s utility from a combination of bids $(b_1, \ldots, b_n)$ is given by:

$$\pi_i(b_1, \ldots, b_n) = \begin{cases} \frac{1}{|\arg\max_j b_j|} - b_i & b_i \in \arg\max_j b_j \\ -b_i & b_i \notin \arg\max_j b_j \end{cases}$$

Our focus is on a symmetric equilibrium—a mixed equilibrium with full support of $[0, 1]$, so that each bidder's bid is distributed in $[0, 1]$ according to the same cumulative distribution function $F$, with the density function $f$ (shown to uniquely exist in [10, 25]). As the bids are distributed in a continuous range, with a non-atomic distribution, we do not need to address cases of ties between them.

When there are no colluders, the setting is similar to one considered in [10, 17, 14], where various results on behaviour of non-cooperative bidders have been provided. To enable us to evaluate the effect of mergers and collusion on the auction, we build on this previous analysis, which we briefly overview in subsections 2.1 and 2.2.

## 2.1 Bids

With no colluders at all, the expected utility of any participant with a bid $b$ is:

$$\pi(b) = (1 - b) \cdot Pr(winning|b) + (-b) \cdot Pr(losing|b)$$

where $Pr(winning|b)$ and $Pr(losing|b)$ are the probabilities of winning or losing the item when bidding $b$, respectively. In a symmetric equilibrium with $n$ players, each of the bidders chooses his bid from a single bid distribution with a probability density function $f_n(x)$ and a cumulative distribution function $F_n(x)$. A player who bids $b$ can only win if all the other $n - 1$ players bid at most $b$, which occurs with probability $F_n^{n-1}(b)$. Thus, the expected utility of a player bidding $b$ is given by:

$$\pi(b) = (1 - b)\, F_n^{n-1}(b) - b\left(1 - F_n^{n-1}(b)\right) = F_n^{n-1}(b) - b$$

In a mixed Nash equilibrium, all points in the support yield the same expected utility to a player, so we have $\pi(x) = \pi(y)$ for all $x, y$ in the support. For an equilibrium with full support, this yields $\pi(0) = \pi(x)$ for all $x \in [0, 1]$. Since $\pi(0) = 0$, this means that for all bids, $F_n^{n-1}(b) = b$. Hence, we have $\left(\int_0^b f_n(x)\mathrm{d}x\right)^{n-1} = b$, implying that $F_n(x) = x^{\frac{1}{n-1}}$

and $f_n(x) = \frac{x^{\frac{2-n}{n-1}}}{n-1}$. Therefore, the expected bid is:

$$E(bid) = \int_0^1 x \cdot \frac{x^{\frac{2-n}{n-1}}}{n-1}\,\mathrm{d}x = \frac{1}{n-1}\int_0^1 x^{\frac{1}{n-1}}\,\mathrm{d}x = \frac{1}{n-1}\cdot\frac{n-1}{n}x^{\frac{n}{n-1}}\Big|_0^1 = \frac{1}{n}$$

The bid's variance is thus:

$$Var(bid) = \int_0^1 x^2 \cdot \frac{x^{\frac{2-n}{n-1}}}{n-1}\,\mathrm{d}x - \frac{1}{n^2} = \frac{1}{2n-1} - \frac{1}{n^2}$$

That is, both the expected bid and the bid's variance monotonically decrease with $n$.

## 2.2 Profits

Given the expected bids, we now more closely examine the profits of all parties. A bidder's profit (BP) is characterised by the probabilistic density function (p.d.f.) $g_{BP}$ below:

$$g_{BP}(z) = \begin{cases} f_n(1-z)F_n^{n-1}(1-z) & z > 0 \\ f_n(-z)(1-F_n^{n-1}(-z)) & z \leq 0 \end{cases}$$

This gives the expected bidder's profit of

$$E(BP) = \int_{-1}^0 z\frac{1}{n-1}(-z)^{\frac{2-n}{n-1}}(1+z)\,\mathrm{d}z + \int_0^1 z\frac{1}{n-1}(1-z)^{\frac{2-n}{n-1}}(1-z)\,\mathrm{d}z = -\frac{1}{n} + \frac{1}{2n-1} - \frac{1-n}{n(2n-1)} = 0$$

The bidders' profit thus has the following variance:

$$Var(BP) = \int_{-1}^0 z^2\frac{1}{n-1}(-z)^{\frac{2-n}{n-1}}(1+z)\,\mathrm{d}z + \int_0^1 z^2\frac{1}{n-1}z^{\frac{2-n}{n-1}}(1-z)\,\mathrm{d}z = \frac{3n^2-5n+2}{n(2n-1)(3n-2)}$$

Differentiating the above gives $\frac{-2n^2+4n-1}{(1-2n)^2n^2}$, which is negative for all $n \geq 2$, so the variance in the bidders' profit decreases as the number of bidders increases.

As the expected profit of all bidders is 0, the auctioneer's profit (AP) is equal to the total social welfare of the auction. In the sum-profit model the auctioneer retains all the bids so his expected profit is simply the sum of expected bids:

$$E(AP) = \sum_{i=1}^n E(bid) = \sum_{i=1}^n \frac{1}{n} = 1$$

In this case, the variance in the auctioneer's utility equals $Var(AP) = \frac{n}{2n-1} - \frac{1}{n}$, monotonically increasing in $n$.

In contrast, in the max-profit model the auctioneer's utility is only the maximal bid, which has the following cumulative distribution function (c.d.f.) $G_{AP}$ :

$$G_{AP}(z) = \begin{cases} F_n^n(z) = z^{\frac{n}{n-1}} & z > 0 \\ 0 & z \leq 0 \end{cases}$$

The expected profit is then given by:

$$E(AP) = \int_0^1 \left(1 - z^{\frac{n}{n-1}}\right)\mathrm{d}z = \frac{n}{2n-1}$$

This expression is monotonically decreasing in $n$. Notice that this value always exceeds $\frac{1}{2}$, so the auctioneer expects to receive more utility from the auction than the utility obtained by all the winners together (as the total value of the item is 1). To find the variance we note that $E(AP^2) = \frac{n}{3n-2}$. Thus, the variance for the max-profit auctioneer is:

$$Var(AP) = \frac{n}{3n-2} - \left(\frac{n}{2n-1}\right)^2 = \frac{n(n-1)^2}{(3n-2)(2n-1)^2}$$

This expression increases with $n$, as we had in the case with the sum-profit auctioneer.

# 3. MERGERS: EQUILIBRIUM ANALYSIS

Consider the case where $k$ out of $n$ bidders work together, and do not hide the fact from the remaining participants that they are cooperating. These bidders coordinate their behaviour, and can thus be thought of as a single player, whose strategy space is the cartesian product of the strategy space of the coordinating agents. We refer to this player as the "merged player" representing the coalition of coordinating players, and refer to the remaining non-merged players as the "singleton players". In *equilibrium*, the joint player best responds to the strategies of the non-merged players, and the strategy of each singleton player is the best response to the other singleton players and the merged player.

As we noted earlier, only a best submission wins the auction, so the merged player would only waste effort if the agents composing it were to make more than one submission. Therefore, since the merging players would only make a single submission (using one of the identities of the merging players), we may consider the joint player as a single bidder, and examine the equilibrium in the resulting game.

The utility of the "merged" player follows that of a single bidder. Therefore, we are essentially seeking a Nash equilibrium for $n-k+1$ bidders, as the merging group would bid using a mixed strategy with full support. The resulting equilibrium is thus equivalent to the equilibrium of the auction with no mergers, but with fewer players. It follows immediately from the analysis of the setting with no mergers that the expected profit for each bidder in this equilibrium would remain zero, that the variance would grow (as it is monotonically decreasing in the number of bidders), and that the expected bid would grow to $\frac{1}{n-k+1}$. In broad terms, this means that for non-merging players, if they win the auction, they would have a lower utility than previously, and if they lose, they would lose more (as the bids get higher). However, the chances of winning do increase, due to the lower number of "actual" participants.

The auctioneer's profit in the sum-profit model would not change, as the total sum of bids is still 1. However, in the max-profit model, the auctioneer's expected profit (and the social welfare) will increase, while its variance will drop.

# 4. COLLUSION

We now analyse the setting with colluding bidders, where the other players are not aware of their collaboration. We first focus on the case with a fixed number of colluders, and then show how the utility of each member depends on the size of a coalition. Finally, we examine the effect of collusion on the profit of the auctioneer and the social welfare.

We wish to emphasise that the analysis of collusion is *by definition* a short-term analysis rather than an analysis of player behaviour in equilibrium. In a merger of players in an auction (be it an all-pay auction or any other auction), the collaborating players obtain an unfair advantage by coordinating their bids, but the fact that they are collaborating is known to the other players. In contrast, the unfair advantage of colluding players stems from the fact that other players are *unaware* of this cooperation; in other words, the non-colluders are *not* best-responding to the colluders' bids.

Our focus in this section is thus on what colluders can achieve in the *short term*, while the non-colluders have not yet figured out that the colluders are working together. In this short term, the non-colluders believe that all the bidders are operating independently, so they expect all the bidders to behave according to the symmetric equilibrium. The colluders can capitalise on this behaviour of the non-colluders, and improve their utility. As the interaction repeats in further auctions, more and more non-colluders may become aware of the agreement among the colluders. We consider possible *middle term* reactions for the non-colluders as they become aware of collusion in Section 5. Once all non-colluders learn of the agreement between the collaborating agents, the system converges to the *long term* equilibrium that characterises the situation where the agreement between the colluders is common knowledge—the equilibrium under mergers, which we already examined in Section 3.

## 4.1 Fixed Number Of Colluders

Suppose we have $k$ colluders out of $n$ bidders, and that the remaining players are not aware of the collusion. Under our assumptions regarding the short-term impact of collusion, we expect the other bidders to play according to the symmetric equilibrium. When the colluders submit a single bid $b$, they win if all the other $n-k$ bids are at most $b$ and lose otherwise, so their total utility is:

$$\pi(b) = (1-b)F_n^{n-k}(b) - b(1-F_n^{n-k}(b)) = b^{\frac{n-k}{n-1}} - b$$

The variance for a fixed bid $b$ is:

$$\begin{aligned} Var(b) &= (1-b-(b^{\frac{n-k}{n-1}}-b))^2 F_n^{n-k}(b) + \\ &+ (-b-(b^{\frac{n-k}{n-1}}-b))^2(1-F_n^{n-k}(b)) = \\ &= b^{\frac{n-k}{n-1}} - b^{\frac{2(n-k)}{n-1}} \end{aligned}$$

To find optimal utility we examine bid when derivative is 0:

$$\pi'(b) = \frac{n-k}{n-1} b^{\frac{1-k}{n-1}} - 1 = 0$$

This implies $b^{\frac{1-k}{n-1}} = \frac{n-1}{n-k}$, yielding the optimal bid:

$$b^* = \left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}$$

To see how this may affect the profits of participants in the auction, consider the following example.

Example 1. *Recall that in the case with no cooperation, there is a mixed Nash equilibrium in which each player bids according to a p.d.f. $f_n(x) = \frac{x^{\frac{2-n}{n-1}}}{n-1}$, with an expected bid of $\frac{1}{n}$ and an expected utility of* 0. *A sum-profit auctioneer thus obtains the expected utility of* 1*, while a max-profit auctioneer gains $\frac{n}{2n-1}$.*

*Thus, in the auction with two bidders, the equilibrium bids are withdrawn from the uniform distribution with a p.d.f. $f_2(x) = 1$, and the expected bid is $\frac{1}{2}$. If there are 3 participants, the bids are distributed according to the c.d.f. $F_3(x) = \sqrt{x}$, with the expected bid of $\frac{1}{3}$. The auctioneer's expected profit in the sum-profit model is* 1*, while in the max-profit model with 3 bidders it is $\frac{3}{5}$.*

*Now, if 2 of these 3 bidders collude, our results show they should bid $\frac{1}{4}$, which gives the colluders the expected profit of $\frac{1}{4}$, while an outsider has the expected loss of $\frac{1}{6}$. The principal's expected profit decreases in both models: in the sum-profit model it drops from* 1 *to $\frac{7}{12}$, and in the max-profit model, from $\frac{3}{5}$ to $\frac{1}{4}F_3(\frac{1}{4}) + \int_{\frac{1}{4}}^{1} b f_3(b)\,\mathrm{d}b = \frac{10}{24} < \frac{3}{5}$.*

It is quite intuitive that the fewer the bidders that are left outside the coalition, the easier it is for colluders to out-bid them. We formally show this.

Lemma 2. *The colluders' optimal bid monotonically decreases with $k$, and monotonically increases with $n$, up to $\frac{1}{e}$.*

Proof. We have to show that the derivatives of $b^*(\cdot)$ with respect to $k$ and $n$ are negative and positive, respectively. We have:

$$(b^*(k))' = -\frac{\left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}}{(k-1)^2}\left((n-k)\ln\left(\frac{n-k}{n-1}\right) + k - 1\right)$$

For any $1 \le k < n$, the first multiplicative term is positive, we only need to examine the sign of the second term, and so it suffices to show that $(n-k)\ln\left(\frac{n-k}{n-1}\right) + k - 1 > 0$. Using the standard logarithm inequality $\ln(1+z) \ge \frac{z}{1+z}$, we obtain the required result:

$$(n-k)\ln\left(\frac{n-k}{n-1}\right) + k - 1 \ge (n-k)\frac{1-k}{n-k} + k - 1 = 0$$

Now, differentiating w.r.t. $n$, we get

$$(b^*(n))' = \frac{\left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}(n-k)\ln\left(\frac{n-k}{n-1}\right) + k - 1}{(n-k)(k-1)}$$

By the same inequality as above,

$$(b^*(n))' \ge \frac{\left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}(k-1) + k - 1}{(n-k)(k-1)} = \frac{1 - \left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}}{n-k} > 0$$

for any $1 \le k < n$, as required.

Finally, we rewrite $b^*$ as $\left(\left(1 - \frac{k-1}{n-1}\right)^{n-1}\right)^{\frac{1}{k-1}}$ and note that for a fixed $k$ and $n \to \infty$, we have that $b^* \to \left(e^{-(k-1)}\right)^{\frac{1}{k-1}} = e^{-1}$. This completes the proof. □

However, even if the collaborators optimise their bid accordingly, the number of outsiders still has a negative effect on their expected utility.

Lemma 3. *The colluders' expected profit under the optimal bid decreases with $n$ and increases with $k$.*

PROOF. The overall expected profit for colluders when bidding optimally is:

$$\pi(b^*) = (b^*)^{\frac{n-k}{n-1}} - b^* = \left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}} \cdot \frac{k-1}{n-1}$$

Differentiating this w.r.t. $n$ gives

$$\frac{\left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}} \ln\left(\frac{n-k}{n-1}\right)}{n-1}$$

which is negative as the first multiplicative term in the numerator is positive, and the logarithm of $\frac{n-k}{n-1} < 1$ is negative. Thus, the profit is monotonically decreasing in $n$. Now, taking the derivative w.r.t. $k$ results in

$$\frac{\left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}} \left(-\ln\left(\frac{n-k}{n-1}\right)\right)}{k-1}$$

This expression is positive using the same argument as before, and so the total expected profit of colluders increases with their number, $k$. □

## 4.2 Optimal Number of Colluders

We now show that not only does colluders' overall utility increase with their group size, individual share of each member also grows. This convexity implies colluders have strong incentives to invite more players to participate in collusion.

THEOREM 4. *The expected profit per colluder increases with $k$.*

PROOF. The individual expected profit for each member of the coalition is:

$$h(k) = \frac{\pi(b^*)}{k} = \left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}} \cdot \frac{k-1}{k(n-1)}$$

The derivative w.r.t. $k$ is given by:

$$h'(k) = -\frac{(n-1)\left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}} \left(k(n-1)\ln\left(\frac{n-k}{n-1}\right) + (k-1)^2\right)}{(k-1)k^2}$$

It suffices to show that the last multiplicative term in the numerator is negative, or, equivalently, that:

$$\ln\left(\frac{n-k}{n-1}\right) < -\frac{(k-1)^2}{k(n-1)}$$

To this end, we use the standard logarithm inequality $\ln(1+x) \leq x$. As required we have:

$$\ln\left(\frac{n-k}{n-1}\right) = \ln\left(1 + \frac{1-k}{n-1}\right) \leq \frac{1-k}{n-1} < -\frac{(k-1)^2}{k(n-1)} \quad \square$$

Hence, the colluders would seek to increase their numbers as much as possible. Next, we explore the effect of collusion on the auctioneer's profit and the social welfare. We show that this effect can be either positive or negative, depending on the number of colluders and the total number of bidders.

## 4.3 Auctioneer's Profits

We now show that if the total number of auction participants is large enough, collusion may be beneficial to the principal in both the sum-profit model and the max-profit model.

THEOREM 5. *In the setting with $k$ colluders, the expected auctioneer utility is $\frac{n-k}{n} + \left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}$ in the sum-profit model and $\frac{n-k}{2n-k-1}\left(1 + \left(\frac{n-k}{n-1}\right)^{\frac{2(n-k)}{k-1}}\right)$ in the max-profit model. The profit in both models decreases in the number of colluders and increases in the total number of participants. For sufficiently large $n$, they exceed the corresponding auctioneer's utilities in the setting without collusion.*

PROOF. The expected profit of a sum-profit auctioneer is given by replacing the bids of $\frac{1}{n}$ for each of the $k$ colluders with a joint single bid of $\left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}$. This results in a total bid sum of $\frac{n-k}{n} + \left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}$. Thus, collusion is obviously profitable for the principal whenever the colluders' bid is larger than $\frac{k}{n}$. This, broadly speaking, is common for smaller $k$ and larger $n$ (as, by Lemma 2, the bid increases with $n$ and decreases with $k$).

For the max-profit model, we examine the maximal bid's distribution, defined by the c.d.f. $G_{AP}$ as follows:

$$G_{AP}(z) = \begin{cases} 0 & z < \frac{n-k}{n-1}^{\frac{n-1}{k-1}} \\ \left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}} & z = \frac{n-k}{n-1}^{\frac{n-1}{k-1}} \\ z^{\frac{n-k}{n-1}} & z > \frac{n-k}{n-1}^{\frac{n-1}{k-1}} \end{cases}$$

Where $G_{AP}(z)$ is not constant, its derivative is $\frac{n-k}{n-1} z^{\frac{1-k}{n-1}}$, so the expected auctioneer's profit is:

$$E(AP) = \left(\frac{n-k}{n-1}\right)^{\frac{2n-k-1}{k-1}} + \int_{\left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}}^{1} \frac{n-k}{n-1} z^{\frac{n-k}{n-1}} \, dz =$$

$$= \frac{n-k}{2n-k-1}\left(1 + \left(\frac{n-k}{n-1}\right)^{\frac{2(n-k)}{k-1}}\right)$$

We compare this with the expected principal's profit in the case of no collusion. To do so, we rewrite it as follows:

$$\frac{n-k}{2n-k-1} + \frac{(n-1)^2}{(2n-k-1)(n-k)}\left(\left(1 - \frac{k-1}{n-1}\right)^{n-1}\right)^{\frac{2}{k-1}}$$

The value of the above expression wobbles for low $n$ and $k$, but for a fixed $k$ and increasing $n$ (i.e., $n \to \infty$) it approaches $\frac{n-k}{2n-k-1} + \frac{(n-1)^2}{(2n-k-1)(n-k)} e^{-2}$. That is, while the expected profit without collusion is edging close to $\frac{1}{2}$, with colluders, the profit is closing in on $\left(\frac{1}{2} + \frac{1}{2e^2}\right)$, and there exists a number of participants $n$ for which the profit with collusion is strictly greater than without collusion. For a fixed $n$, the profit is monotonically decreasing in $k$, which fits with earlier results indicating a lower bid from colluders, as their cohort grows. □

## 4.4 Social Welfare

We now analyse social welfare in the setting with colluders. To this end, we need to calculate the expected profits of the non-colluders. Surprisingly, as Theorem 6 below shows, in some cases they may even benefit from other players colluding. Overall, however, the presence of colluders does not affect the social welfare in the sum-profit model, and may have either a positive or a negative effect in the max-profit model, depending on the parameters of the setting.

Theorem 6. *The social welfare in the sum-profit model does not change due to collusion. In the max-profit model, the presence of colluders may have different effects on the social welfare, depending on the relation between the number of colluders and the total number of participants. In particular, the social welfare drops for settings with many participants.*

Proof. We calculate expected profit for non-colluders, defined by p.d.f. $g$ below, depending on colluders' bid $b^*$:

$$g(z) = \begin{cases} f(-z)(1 - F_n^{n-k}(-z)) & -1 \geq z < -b^*(k) \\ f(-z) & -b^*(k) \leq z \leq 0 \\ f(1-z)F_n^{n-k}(1-z) & 0 < z < 1 - b^*(k) \end{cases}$$

The expected profit of a non-colluder agent is: $E(z) = \int_{-1}^{-b^*} \frac{z}{n-1}(-z)^{\frac{2-n}{n-1}} \left(1 - (-z)^{\frac{n-k-1}{n-1}}\right) \mathrm{d}z + \int_{-b^*}^{0} \frac{z}{n-1}(-z)^{\frac{2-n}{n-1}} \mathrm{d}z + \int_{0}^{1-b^*} \frac{z}{n-1}(1-z)^{\frac{2-n}{n-1}}(1-z)^{\frac{n-k-1}{n-1}} \mathrm{d}z = \frac{k-n(b^*)^{\frac{n-k}{n-1}}}{n(n-k)}$.

This expression may be positive or negative, depending on $k$ and $n$. As the colluders' bid does not exceed $\frac{1}{e}$, for small $k$ the expression takes a negative value. However, when $k$ is rather large with regard to $n$ (e.g., when $k$ is roughly $\frac{n}{2}$), it is positive. That is, the non-colluders may benefit from collusion, despite not being aware of it.

Summing up the expected profits of all the parties in the sum-profit model results in the same social welfare as in the case of mergers or of no bidder cooperation:

$$\frac{k}{n} - \frac{n-k}{n-1}^{\frac{n-k}{k-1}} + \frac{n-k}{n-1}^{\frac{n-k}{k-1}} - \frac{n-k}{n-1}^{\frac{n-1}{k-1}} + \frac{n-k}{n} + \frac{n-k}{n-1}^{\frac{n-1}{k-1}} = 1$$

However, in the max-profit model the results are more ambiguous. For very large $n$, looking coarsely at the non-colluders' expected profit, we see that when we have $n-k$ such players, the sum of their expected losses approaches $-\frac{1}{e}$. We already know that in this scenario the expected profit of colluders is 0, so we need to examine this in relation to the changes in the profits of the auctioneer (see Section 4.3). In this case, the auctioneer's profit approaches $\frac{1}{2} + \frac{1}{2e^2}$, so the social welfare drops below $\frac{1}{2}$, which is lower than what would happen without colluders. □

# 5. RESPONSE TO COLLUSION

Section 4 examined the short-term collusion impact, where non-colluders continue behaving as the symmetric equilibrium prescribes (which is sub-optimal in the presence of colluders). This approach is justified by the fact that this symmetric equilibrium is a mixed one, so collusion may be difficult to detect. However, after many interactions, a non-colluder may notice that his winning rate is different from what he would expect under the symmetric equilibrium, and suspect foul play. How would he respond to the colluding coalition? Would it make sense to collaborate with other participants and play jointly against the colluders? In this section we consider two scenarios for the middle-term: where there exists a single player who is aware of collusion, and where several colluding coalitions are possible.

## 5.1 A Player Aware of Collusion

If one of the players becomes aware of $k$ other bidders colluding, he would never submit a non-zero bid below the colluders' bid $b^*(k)$, as then he would lose and get a negative utility. He would rather respond by either bidding 0 (thus obtaining zero utility), or placing a bid $b$ which is higher than $b^*(k)$. In the latter case, his expected profit is:

$$\pi(b) = (1-b)F_n^{n-k-1}(b) - b(1 - F_n^{n-k-1}(b)) = b^{\frac{n-k-1}{n-1}} - b$$

Note that this is always positive (as $b \leq 1$ and $\frac{n-k-1}{n-1} < 1$), hence it is always beneficial to bid above $b^*(k)$ rather than 0. As the optimal bid for $k+1$ colluders is $b^*(k+1)$, which according to Lemma 2 is smaller than $b^*(k)$, the best bid for the responder is the smallest possible value above $b^*(k)$. Having a larger bid is less profitable, as the bidder's expected profit is monotonically decreasing in $k$ when $k > b^*(k)$.

Now, since this bid is larger than $\frac{1}{n}$, this means that in the sum-profit model, it is beneficial for the auctioneer to expose the existence of a collusion ring to some players. Similarly, in the max-profit model, the expected profit grows (especially when $k$ is significantly smaller than $n$), making the revelation profitable to this auctioneer type as well.

## 5.2 Several Groups Of Colluders

If there are several groups of colluders that are not aware of one another, each would bid its optimal value as prescribed by the previous analysis. By Lemma 2, this bid decreases with the size of a coalition, and so the smallest coalition would outbid the others and get positive (though sub-optimal) expected utility. Indeed, suppose there are $m$ colluder groups, each with $k_i$ colluders, and let $k_{min} = \min\{k_1, k_2, \ldots, k_m\}$. The expected profit of the smallest (winning) coalition is

$$\left(\frac{n-k_{min}}{n-1}\right)^{\frac{n-1}{k_{min}-1} \cdot \frac{n-\sum_{i=1}^{m} k_i + m - 1}{n-1}} - \left(\frac{n-k_{min}}{n-1}\right)^{\frac{n-1}{k_{min}-1}}$$

$$= \left(\frac{n-k_{min}}{n-1}\right)^{\frac{n-\sum_{i=1}^{m} k_i + m - 1}{k_{min}-1}} - \left(\frac{n-k_{min}}{n-1}\right)^{\frac{n-1}{k_{min}-1}}$$

As the bids are larger with a smaller number of colluders, the auctioneer prefers several small groups of colluders over a single big one. If $k_{min}$ is large enough, it becomes worthwhile for a sum-profit auctioneer to uncover collusion rings and publicise them. Similarly, for a max-profit auctioneer, it may be worthwhile to expose the collusions (even more so than for the sum-profit one). For example, for $n > 6$, the sum-profit auctioneer would rather divide the bidders into pairs. However, for the max-profit auctioneer, it is never profitable to have all bidders be colluders. As the maximal bid of colluders is $\frac{1}{e}$ (by Lemma 2), and his expected profit without them is above $\frac{1}{2}$, the max-profit auctioneer "needs" non-colluders to increase his expected profit.

# 6. RELATED WORK

Research into all-pay auctions originates in political science, dealing with lobbying [18, 9], but much of the analysis is found in auction theory studies [22, 20].[3] When bidders have the same value distribution for the item, [25] showed that there is a symmetric equilibrium in auctions where the winner is the bidder with highest bid. A prominent study of all-pay auctions in full information settings is [10], showing how most valuations—apart from the top two—are irrelevant to the analysis of the winner, and (using [18]), showing that in most cases, the possible equilibria are those with full support on the range from 0 to the second highest valuation; when

[3]Such collusion is also somewhat reminiscent of work on political mergers [28, 5, 3].

all players have the same valuation, a full support is the only symmetric equilibria possible. That work does not deal with cooperation among players, but helps validate our choice of focusing on full-support equilibria. Collusions in auctions (nicknamed "bidding rings") was examined for various auction types [26, 4, 6, 15, 7, 2]. First-price and second-price auctions differ in their ability to "self-police" each bid [22, 20, 24], which affects the impact of collusion in them. A model of mergers with full information was proposed in [19] for many auctions, including auctions where each bidder makes an investment, and gains are divided among the bidders. It explores various auction models in which there is no single winner, but rather profits are distributed among the players according to their investment relative to the others. It shows that mergers, in many domains (e.g., when there is a marked benefit to be the top bidder), are profitable for their participants, while when there is not a significant benefit to being the maximal bidder, mergers may still be beneficial; however, players do not coalesce around a single bid, but rather divide their resources among them. In this setting the only first-price auction is an all-pay one, and due to its full information assumptions, that work dismisses all-pay auctions as uninteresting, since the bidders' profit is always 0.

A recent paper models auction collusion using voting techniques [21]. There, bidders prefer some adversaries winning over others, and so collude with them. Work on partial information in auction collusions is somewhat limited and focuses on the principal's ignorance rather than on bidders hiding information from one another (e.g. [16, 1]). It neither deals with all-pay auctions, nor with rival groups of colluders.

# 7. SUMMARY AND DISCUSSION

All-pay auctions are an important domain in which mergers and collusion can have a strong impact. Indeed, bidders in an all-pay auction are somewhat weak, as they lose their bid irrespective of whether or not they obtain the item. In this paper, we showed that they can increase their power, and improve social welfare, by collaborating with one another. Our technical results are summarised in Table 1. We demonstrated that mergers have a small positive effect on social welfare (which for $n \to \infty$ actually converges to the same value as in a non-cooperative auction), and only the auctioneer benefits in expectation (bidders remain with an expected utility of 0). In contrast, collusion may lower social welfare below the non-cooperative case, but makes the auction more egalitarian: the colluders, and not just the auctioneer, obtain a positive profit in expectation.

Now, not only do these properties help the bidders—the lack of transparent information enables knowledgable players, as well as the auctioneer, to effectively manipulate the auction to their benefit. For instance, auctioneers would rather have small collusion rings. We identified the values for which auctioneers may choose to reveal a collusion ring, making it effectively a merger, so as to increase their utility.

As an example, consider the sum-profit auctioneer who strives to have a large number of small collusion rings, ideally several pairs, but who would be satisfied with a larger number of colluders up to a certain limit (for large enough $n$, approximately $\frac{n}{e}$). In a sense, the auctioneer is implementing a "divide and conquer" strategy on the bidders, as he strives to keep them small and separate. On the other hand, a max-profit auctioneer would like a small colluding group, but wishes to have a significant number of players be non-cooperative. This auctioneer makes use of the colluders as a sort of "insurance"—with a large number of players, variation gets smaller, and probability of a large bid falls. However, colluders (and the fewer the better) give the auctioneer a significant minimum-bid, on which he can rely.

Our approach in this paper has some limitations. First, our model is a simplified version of the all-pay auction of Baye et al. [10], where the participant with the maximal effort always wins the auction. More realistic models have a probabilistic relation between the effort levels and winning, such as the Tulloc contest function (see [27, 10, 14]). Also, our model is simplistic in that all participants are symmetric and equal in their ability or skill, so for each of them the same effort results in the same quality. Beyond the simplicity of our model, our analysis focuses on the long-term of collaboration, where the agreement becomes common knowledge, or on the short-term, where the non-colluders act naively and follow the sub-optimal mixed equilibrium strategy. We only briefly touched on the middle-term, where agents have partial information regarding the coalitions of other bidders. A detailed analysis of the middle-term dynamics is required in order to have a complete picture of the impact of bidder collaboration in all-pay auctions. Our analysis further assumes a constant number of bidders, which is common knowledge. This is not realistic in many anonymous settings, where bidders may freely come and go.

Finally, we analyse only one contest structure, where there is a simultaneous interaction among all the agents. Other research examines alternative structures like tournaments, or contests with multiple winners [17], and the impact of collusion in such settings should also be studied.[4]

Several questions remain open for future research. First, how would our results change in more general settings, such as domains when players are assumed to have heterogeneous skill levels, or where the contest structure is richer? How can bidders detect collusion by other players, and how many repeated interactions would they need to do so? Can our theoretical results be corroborated by empirical evidence from real-world all-pay auctions? Finally, (how) can crowdsourcing contests be designed to be resistant to collusion?

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[4] We have examined the dataset of [17] which recorded the quality levels of submissions to an advertising crowdsourcing contest. There is a wide quality gap between the best and second-best submissions in the vast majority of these auctions, indicating that collusion could allow agents to win the same reward with significantly less effort.

| Variable | No cooperation | Mergers | Collusion |
|---|---|---|---|
| Expected bid<br>[Variance] | $\frac{1}{n}$<br>$\left[\frac{1}{2n-1} - \frac{1}{n^2}\right]$ | $\frac{1}{n-k+1} \uparrow$<br>$\left[\frac{1}{2(n-k+1)-1} - \frac{1}{(n-k+1)^2}\right] \uparrow$ | colluders: $\left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}} \downarrow [0]$<br>non-colluders: $\frac{1}{n} \left[\frac{1}{2n-1} - \frac{1}{n^2}\right]$ |
| Bidder utility<br>[Variance] | $0$<br>$\left[\frac{3n^2-5n+2}{n(2n-1)(3n-2)}\right]$ | $0$<br>$\left[\frac{3(n-k+1)^2-5(n-k+1)+2}{(n-k+1)(2(n-k+1)-1)(3(n-k+1)-2)}\right] \uparrow$ | colluders: $\left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}} \left(\frac{k-1}{n-1}\right) \uparrow$<br>$\left[\left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}} - \left(\frac{n-k}{n-1}\right)^{\frac{2(n-k)}{k-1}}\right]$<br>non-colluders: $\frac{k-n\left(\frac{n-k}{n-1}\right)^{\frac{n-k}{k-1}}}{n(n-k)}$ |
| Sum-profit principal utility<br>[Variance] | $1$<br>$\left[\frac{n}{2n-1} - \frac{1}{n}\right]$ | $1$<br>$\left[\frac{n-k+1}{2n-2k+1} - \frac{1}{n-k+1}\right] \downarrow$ | $\frac{n-k}{n} + \left(\frac{n-k}{n-1}\right)^{\frac{n-1}{k-1}}$ |
| Max-profit principal utility<br>[Variance] | $\frac{n}{2n-1}$<br>$\left[\frac{n(n-1)^2}{(3n-2)(2n-1)^2}\right]$ | $\frac{n-k+1}{2n-2k+1} \uparrow$<br>$\left[\frac{(n-k+1)(n-k)^2}{(3n-3k+1)(2n-2k+1)^2}\right] \downarrow$ | $\frac{n-k}{2n-k-1}\left(1 + \left(\frac{n-k}{n-1}\right)^{\frac{2(n-k)}{k-1}}\right)$ |

**Table 1: The values, in expectation, of some of the variables in a non-cooperative setting, when $k$ members merged, and when $k$ members are colluding. Arrows indicate monotonicity of expression, as $k$ grows.**


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