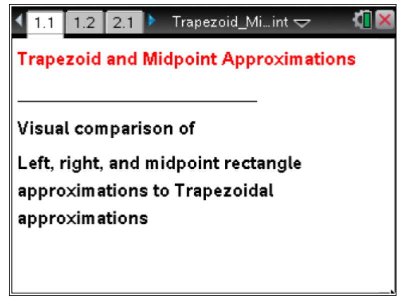
Trapezoid and Midpoint Approximations

MATH NSPIRED

TEACHER NOTES



Math Objectives

- Students will be able to identify whether rectangle or trapezoidal approximations will provide an overestimate or an underestimate of the area under a curve from a positive function graph.
- Students will be able to order the estimates of area under a curve.
- Students will be able to describe the relationship of concavity to the midpoint and trapezoidal approximations.
- Students will look for and make use of structure. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

- trapezoidal approximations
- left, right, and midpoint rectangle approximations
- area under a curve

About the Lesson

- This lesson involves providing students with a visual representation of area estimation methods in order to determine which is most accurate.
- As a result, students will:
  - Observe the left endpoint rectangle, right endpoint rectangle, and trapezoidal estimates of the area under the curve in order to observe and make predictions about the accuracy of each estimate.
  - Experience a dynamic comparison of the midpoint rectangle and trapezoidal estimates of area under a curve, relating these to the concavity of the curve.
  - Suggest their own geometric estimation methods for area under a curve to improve on the accuracy of the midpoint rectangle estimate.

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- Use Quick Poll to assess student understanding.

Activity Materials

- Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

- Trapezoid_Midpoint_Student.pdf
- Trapezoid_Midpoint_Student.doc

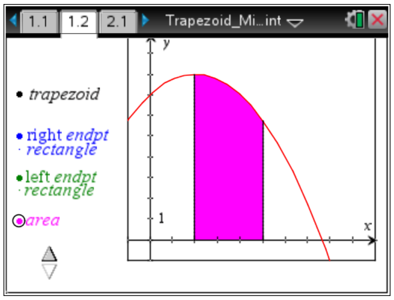
TI-Nspire document

- Trapezoid_Midpoint.tns

Discussion Points and Possible Answers

Tech Tip: Once the minimized slider is selected the students can use the keypad arrows (▲ and ▼) to arrow up and down.

Move to page 1.2.



1. Use the up arrow to view the estimates of the shaded area under the curve.
   a. How is the left endpoint rectangle constructed?

      Answer: It uses the value of the function at the left end of the interval to determine the height of the rectangle. The width of the rectangle is the width of the interval.

   b. How is the right endpoint rectangle constructed?

      Answer: It uses the value of the function at the right end of the interval to determine the height of the rectangle. The width of the rectangle is the width of the interval.

   c. How is the trapezoid constructed?

      Answer: The length of one base of the trapezoid is the value of the function at the left end of the interval. The length of the other base of the trapezoid is the value of the function at the right end of the interval. The height of the trapezoid is the width of the interval.

2. Which of these three estimates in question 1 overestimates the actual area? Which of these underestimates? Which do you think is the closest estimate?

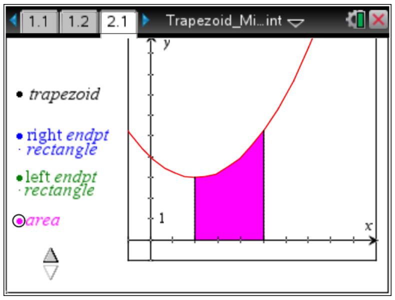
   Answer: The left endpoint rectangle is an overestimate; the right endpoint rectangle and the trapezoid are both underestimates. The trapezoid is the closest estimate.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 1 at the end of this lesson.

3. Will your answers to question 2 be true for any function? Explain your reasoning.

Sample answer: Student answers may vary. The answers will not be true for every function. If the function is increasing over the interval on which you are estimating the area, the left endpoint rectangle will be an underestimate, while the right endpoint rectangle will be an overestimate. Whether the trapezoidal estimate is under or over will depend on the shape of the curve. (It is also possible that one of the rectangle estimates could be closer than the trapezoidal estimate. Think of an increasing curve that is very close to horizontal for most of the interval and then increases rapidly just before the right endpoint. For such a curve, the area of the left endpoint rectangle might well be closer to the true area than the trapezoidal area.)

Move to page 2.1.



4. Use the up arrow to view the estimates of the shaded area under the curve. Is your prediction from question 3 still true? If so, explain why. If so, explain why. If not, how would you adjust your prediction?

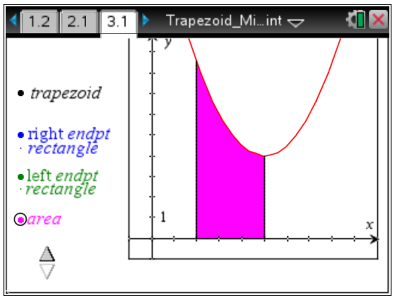
Sample answer: Student answers may vary.
Students may need to adjust incorrect predictions, having not considered increasing functions. The new prediction should reflect the differences in increasing and decreasing functions, as described in the answer to question 3.

5. When will the trapezoid be an overestimate? When will it be an underestimate? When will it be the best estimate? Explain your reasoning.

Answer: The trapezoid could be either an underestimate or an overestimate in either case of a decreasing or increasing function. It will not always be the best estimate (out of the three choices), but it is the same as the average of the areas of the left endpoint and right endpoint rectangles for a positive function. For a linear function, however, the trapezoidal approximation will be exactly correct.

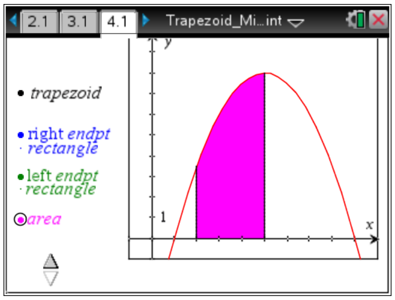
TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

Move to pages 3.1 and 4.1.

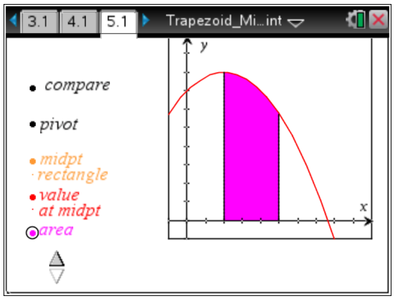


6. Use the up arrow on each page to view the estimates of the area. Was your prediction from question 5 correct? If it is correct, explain why. If it is incorrect, explain how you would adjust your prediction to reflect your observations.

   Answer: For the decreasing concave up curve shown on page 3.1, the left rectangle and trapezoidal approximations are overestimates and the right rectangle is an underestimate. For the increasing concave down curve shown on page 4.1, the left rectangle and trapezoidal approximations are now underestimates and the right rectangle is an overestimate.



Move to page 5.1.



7. Use the up arrow to view the midpoint rectangle estimate of the area under the curve. (Don't go past the midpoint rectangle yet.)
   a. How is the midpoint rectangle constructed?

      Answer: The height of the rectangle is given by the value of the function at the midpoint of the interval over which area is being estimated. The width of the rectangle is the width of the interval.

   b. Why is it hard to tell whether the midpoint rectangle area is an overestimate or underestimate?

      Sample answer: Answers will vary. Part of the curve lies above the top of the rectangle and part lies below, so it is not obvious whether the midpoint rectangle is an overestimate or underestimate. Students may reason that one part does appear larger than the other, so there is not an exact cancellation.

8. Use the up arrow to pivot.
   a. What are you pivoting when you click the arrow?

      Answer: The top of the rectangle is pivoting about the midpoint.

b. As the segment pivots, how does the shaded area compare to the shaded area of the original midpoint rectangle? How do you know?

Answer: The areas are the same. Pivoting the top of the rectangle doesn't change the overall area, since the area portion above the original is exactly the same as the area portion below the original (two congruent right triangles are formed).

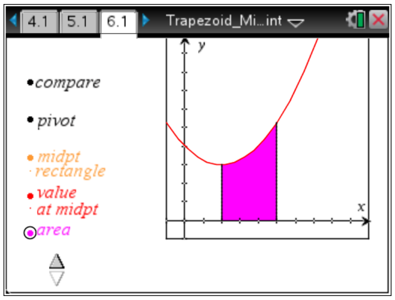
c. Once the segment is done pivoting, a dashed line below appears. What does it represent?

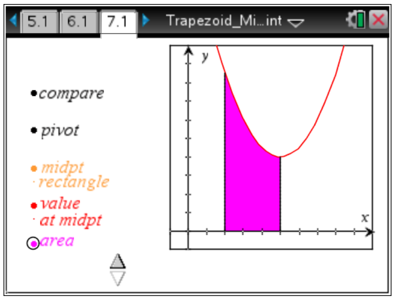
Answer: The dashed line represents the top of the trapezoid that would be used to estimate the area, using the method described earlier in the activity.

d. Is the midpoint rectangle area an overestimate or an underestimate? Explain.

Answer: The trapezoid with its top tangent to the curve clearly has an area greater than that under the curve. You know that its area is the same as the midpoint rectangle, so it must be an overestimate. (The midpoint rectangle appears to provide a slightly better estimate of area than the trapezoidal approximation, but in the opposite direction.)

Move to pages 6.1, 7.1, and 8.1.





9. Like the function graph on page 5.1, these functions are all positive, but they vary as to whether they are increasing or decreasing, or concave up or concave down. Use these to identify which of the rectangle (L, R, or M) and trapezoidal (T) approximations are underestimates and which are overestimates of the definite integral:

a. Function is positive, increasing, and concave up.

Answer: underestimate: L, M overestimate: R, T

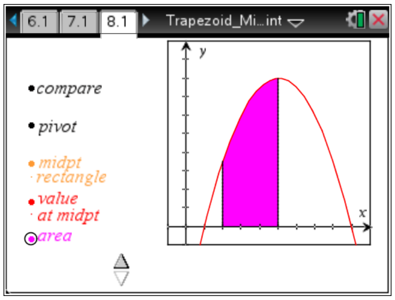
b. Function is positive, decreasing, and concave up.

Answer: underestimate: R, M overestimate: L, T

c. Function is positive, increasing, and concave down.

Answer: underestimate: L, T overestimate: R, M

d. Function is positive, decreasing, and concave down.



Answer: underestimate: R, T overestimate: L, M

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 3 at the end of the lesson.

Teacher Tip: Students can argue that the midpoint rectangle is a better estimate than the trapezoid by comparing the difference between each and the actual area. You may want to encourage students to try this with analytic geometry, using triangles to estimate the area of the excess or deficiencies between each estimate and the curve.

10. How do the answers in question 9 change if *positive* is replaced by *negative*?

Answer: The use of positive functions throughout this activity allows for a direct interpretation in terms of area. When the functions are negative, then that interpretation must be reconsidered for definite integrals. The rectangle approximations can now be thought of as being obtained by replacing the function in the definite integral with very simple (constant) functions. The trapezoidal approximation can be thought of as being obtained by replacing the function with a linear function. Since all the signs are switched, but the area magnitudes are not, the answers for question 9 with negative functions simply have *under* and *over* switched.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 4 at the end of the lesson.

Wrap Up

Upon completion of this discussion, the teacher should ensure that students understand:

- That simple definite integral approximations are based on simple geometric figures using known area formulas (rectangle and trapezoid).
- How monotonic increasing or decreasing behavior orders left, right, and trapezoidal approximations.
- How monotonic concavity affects and orders the midpoint and trapezoidal approximations.

Assessment

Ask students to determine if splitting up the interval into more rectangles provides a better estimate. Is it possible to use more rectangles, but take left-hand rectangles, and get a better estimate than one midpoint rectangle gives?

A good assessment item is to specify an order for the relationship among the three rectangle approximations and the trapezoidal approximation. Then ask students to sketch the graph of a function over an interval for which that ordering would be true. For example, if you are given that $L < M < T < R$, then a function that is positive, increasing, and concave up would have the approximations in that order.

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Note 1

Question 2, *Quick Poll:* Use a *Quick Poll* to determine which of the estimates students think is an overestimate, which is an underestimate, and which is the closest estimate. This could lead to an opportunity for a whole class discussion about whether or not this will always be true.

Note 2

Question 5, *Quick Poll:* Use a *Quick Poll* to determine when students think a trapezoid will be an overestimate, when it will be an underestimate, and when it will be “just right” (exactly correct). This could be used as an entry point for a whole group discussion about the sensitivity of the trapezoidal estimate to concavity.

Note 3

Question 9, *Quick Poll:* Question 9 lends itself to a *Quick Poll* very well.

Note 4

Question 10, *Quick Poll:* Question 10 also lends itself to a *Quick Poll* very well. A whole class discussion could center on resolving how the value of a definite integral takes into account the sign of the function.