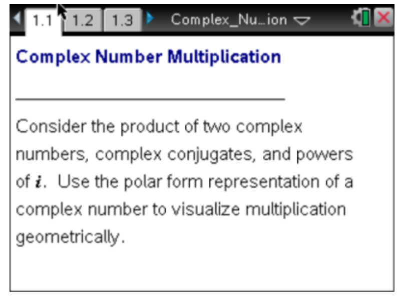


Math Objectives

- Students will compute the product of two complex numbers.
- Students will discover the pattern in powers of the imaginary number i.
- Students will compute the product of complex conjugates and relate this value to the absolute value of the original complex number.
- Students will graph complex numbers in polar form.
- Students will compute the product of complex numbers in polar form.

Vocabulary

- complex conjugate
- absolute value
- magnitude
- argument
- polar form

About the Lesson

- This lesson involves the product of complex numbers, powers of i, and complex conjugates.
- As a result students will:
  - Compute the product of two complex numbers and write a general symbolic result for the product.
  - Discover the pattern in powers of i and simplify i raised to any power.
  - Compute the product of complex conjugates and relate this to the absolute value of the original complex number.
  - Visualize the product of complex numbers written in polar form and write the product in polar form.

TI-Nspire™ Navigator™ System

- Use Class Capture to examine patterns that emerge.
- Use Live Presenter to demonstrate.
- Use Quick Poll to assess students' understanding.

Activity Materials

Compatible TI Technologies : TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Tech Tips:

- This activity includes screen captures from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire Apps. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

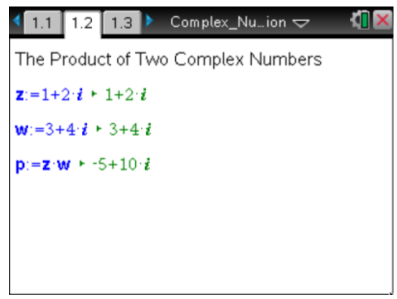
Student Activity
Complex_Number_Multiplication_Student.pdf
Complex_Number_Multiplication_Student.doc

TI-Nspire document
Complex_Number_Multiplication.tns

Discussion Points and Possible Answers

> **Tech Tip:** If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand ( ) getting ready to grab the point. Also, be sure that the word "point" appears, not the word text. Then press [ctrl] [ ] to grab the point and close the hand ( ). There are several interactive Math Boxes in this activity. Remember that if you change a definition in one Math Box, all other interactive Math Boxes related to this change will be automatically updated.

Move to page 1.2.



1. This Notes Page contains three interactive Math Boxes for the complex numbers z and w and the product $p = z \cdot w$.
   a. Redefine z and/or w as necessary to complete the following tables. Note: to redefine z or w, edit the Math Box following the assignment characters, $:=$.

> **Tech Tip:** To access i on the handheld, press [π▸] to obtain a list of mathematical symbols. Use the arrow keys to highlight i and press [enter].

Answer:

z	$1+5i$	$2-3i$	$-2+4i$	$-3-4i$
w	$2+3i$	$3-7i$	$1-2i$	$-2-6i$
$z \cdot w$	$-13+13i$	$-15-23i$	$6+8i$	$-18+26i$

z	$7+7i$	$3-11i$	$2+3i$	$-2-3i$
w	i	$-i$	$2+3i$	$-2-3i$
$z \cdot w$	$-7+7i$	$-11-3i$	$-5+12i$	$-5+12i$

b. Let $z = a + bi$ and $w = c + di$. Write the product, $p = z \cdot w$, symbolically in terms of the constants a, b, c, and d.

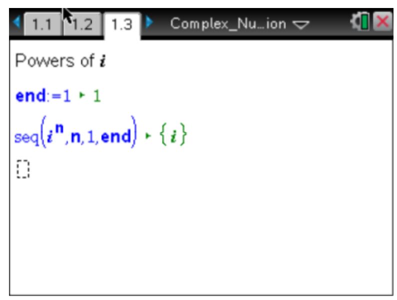
Answer:

$$\begin{aligned} p = z \cdot w &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + bd(-1) + adi + bci \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Note: Students can check this computation using the CAS handheld.

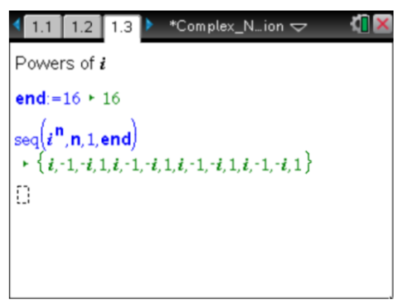
TI-Nspire Navigator Opportunity: *Class Capture*
See Note 1 at the end of this lesson.

Move to page 1.3.



2. This Notes Page contains two interactive Math Boxes to produce a sequence of the powers of $i = \sqrt{-1}$. That is, it constructs a sequence of the form $i, i^2, i^3, \ldots, i^{end}$. The variable **end** is the largest value the sequence variable will assume, in this case the last power of i..

   a. Change the value of the variable **end** as necessary to complete the following tables. Note: the variable **end** is defined in a Math Box. To redefine the value of **end**, edit the Math Box following the assignment characters, $:=$.



Answer:

n	1	2	3	4	5	6	7	8
i^n	i	-1	$-i$	1	i	-1	$-i$	1

n	9	10	11	12	13	14	15	16
i^n	i	-1	$-i$	1	i	-1	$-i$	1

b. In words, describe the pattern in the powers of i.

Sample Answers: The powers of i repeat in a cycle of length 4 as suggested by the table above. Every fourth power is the same. The cycle is i, -1, $-i$, or 1.

c. Without using your calculator, use your answer in part 2b to find i^{25} and i^{103}.

Answer:

$i^{25} = i^{24} \cdot i^{1} = \left(i^{4}\right)^{6} \cdot i^{1} = 1 \cdot i = i$

Or $25 \div 4 = 6$ with remainder 1. Therefore, $i^{25} = i^{1} = i$.

$i^{103} = i^{100} \cdot i^{3} = \left(i^{4}\right)^{25} \cdot i^{3} = 1 \cdot i^{3} = -i$

Or $103 \div 4 = 25$ with remainder 3. Therefore, $i^{103} = i^{3} = -i$.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

Move to page 1.4.

3. This Notes Page contains three interactive Math Boxes for the complex number z, its complex conjugate cz (denoted $\bar{z}$), and the product $z \cdot cz$.

   a. Change the value of z as necessary to complete the following table. To change the value of z, edit the Math Box following the assignment characters, $:=$.

Answer:

z	$1+2i$	$2-3i$	$-3+4i$	$-4-5i$
$\bar{z}$	$1-2i$	$2+3i$	$-3-4i$	$-4+5i$
$z \cdot \bar{z}$	5	13	25	41

b. For $z = a + bi$, find $z \cdot \overline{z}$.

Answer:

$$z \cdot \overline{z} = (a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2$$
$$= a^2 - b^2(-1) = a^2 + b^2$$

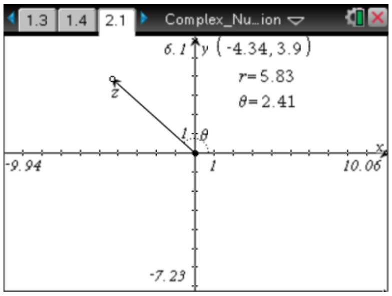
c. Recall that a complex number can be represented by a point in the complex plane. For $z = a + bi$ find $r = \sqrt{z \cdot \overline{z}}$ and interpret this value.

Answer:

$$r = \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + b^2}$$

This value is the absolute value, or magnitude, of the complex number z. The product of complex conjugates z and $\overline{z}$ is the square of the distance from the origin to the point representing z in the complex plane. Note: the product of a complex number and its conjugate is a real number.

Move to page 2.1.



For any complex number $z = a + bi$, the absolute value, or magnitude, is $r = |z| = \sqrt{a^2 + b^2}$. The absolute value is the distance from the origin to the point representing z in the complex plane. The argument of the complex number z, $\arg(z)$, is the angle θ (in radians) formed between the positive real axis and the position vector representing z. The angle is positive if measured counterclockwise from the positive real axis.

4. On Page 2.1, the complex number z is represented by a point and a position vector. The value of z, the absolute value, and the argument are given on this page. Drag and position z as necessary to answer the following questions.

Describe the location of the point representing z in the complex plane if:

a. $r = 2$ and $\theta = \dfrac{\pi}{4}$

Sample Answers: The point representing z lies in the first quadrant, 2 units from the origin, and on the ray from the origin that makes an angle $\dfrac{\pi}{4}$ with the positive real axis.

Note: It might be difficult for students to move the point z in order to obtain the exact values of r and θ. Students will also have to convert the symbolic value of θ to a decimal approximation.

b. $r = 4$ and $\theta = \frac{5\pi}{6}$

Sample Answer: The point representing z lies in the second quadrant, 4 units from the origin, and on the ray from the origin that makes an angle $\frac{5\pi}{6}$ with the positive real axis.

c. $r = 1$ and $\theta = -\frac{4\pi}{3}$

Sample Answer: The point representing z lies in the third quadrant, 1 unit from the origin, and on the ray from the origin that makes an angle $-\frac{4\pi}{3}$ with the positive real axis. Note: this point lies on the unit circle.

d. $r = 3$ and $\theta = \frac{13\pi}{4}$

Sample Answer: The point representing z lies in the third quadrant, 3 units from the origin, and on the ray from the origin that makes an angle $\frac{13\pi}{4}$ or $\frac{5\pi}{4}$ with the positive real axis.

e. $r = 3$ and $\theta = \pi$

Sample Answer: The point representing z lies on the real axis and 3 units to the left of the origin (on the ray from the origin that makes an angle π with the positive real axis).

f. $r = 2$

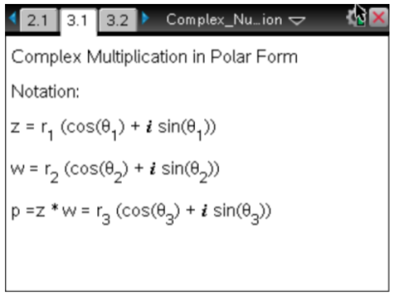
Sample Answer: This equation describes infinitely many points in the complex plane that lie on the circle of radius 2 which is centered at the origin.

g. $\theta = \frac{3\pi}{2}$

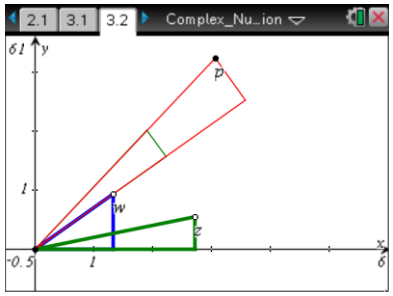
Sample Answer: This equation describes infinitely many points in the complex plane that lie on the ray from the origin that makes an angle $\frac{3\pi}{2}$ with the positive real axis.

TI-Nspire Navigator Opportunity: *Class Capture*
See Note 3 at the end of this lesson.

Move to page 3.2.



Any complex number $z = a + bi$ with $r = |z|$ and $\theta = \arg(z)$ can be written in polar form as $z = r(\cos\theta + i\sin\theta)$. Page 3.2 illustrates the product of two complex numbers in polar form.



5. The complex numbers z, w, and p are represented by triangles. When you drag either the point z or the point w, the product is automatically computed, and the triangle representing p is updated. Note that a copy of the triangle representing z is rotated so that the vertex lies along the hypotenuse of the triangle that represents w. Move z and w around the first quadrant, and observe the absolute value and argument for the three complex numbers.

   a. Write an equation which seems to define θ_3 in terms of θ_1 and θ_2.

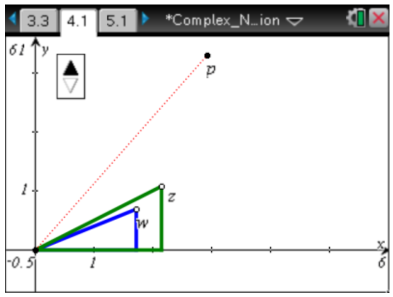
   <u>Sample Answers:</u> The diagram suggests that $\theta_3 = \theta_1 + \theta_2$.

   Note: Students might need a hint to focus on the angles measurements.

   b. Write an equation which seems to define r_3 in terms of r_1 and r_2.

   <u>Answer:</u> The diagram suggests that $r_3 = r_1 \cdot r_2$. That is, the absolute values are multiplied.

Move to page 4.1.

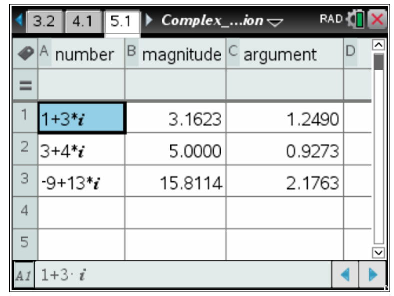


On Page 4.1, click on the arrows to step through the process of multiplication. This figure might provide further insight about the relationship among the absolute values and arguments.

Tech Tip: To animate the figure, select the slider, press ctrl menu, and select **Animate**.

Tech Tip: To animate the figure, tap and hold the slider and select **Animate.**

Move to page 5.1.



6. Use this Lists and Spreadsheet page to test your hypotheses. Consider various values for z and w (cells A1 and A2) in polar form and try to prove your guess.

   <u>Sample Answers:</u> Consider the product of two complex numbers in polar form.

$$z \cdot w = (r_1(\cos\theta_1 + i\sin\theta_1))(r_2(\cos\theta_2 + i\sin\theta_2))$$
$$= r_1r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2)$$
$$= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))$$
$$= r_1r_2(\cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2))$$

   To obtain the product in polar form, multiply the absolute values, and add the arguments.

Extensions

1. For a complex number z in polar form, ask students to write a formula for z^n and $\sqrt[n]{z}$ in polar form.
2. For two complex numbers z and w in polar form, ask students to write a formula for $\frac{z}{w}$ in polar form.
3. Ask students to describe the point(s) representing z in the complex plane if $r < 0$ and/or $\theta < 0$.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- How to multiply two complex numbers in $a + bi$ form.
- How to multiply two complex numbers in polar form.
- The cyclic nature of powers of i.
- The relationship between the product of complex conjugates and the absolute value of the original complex number.

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Note 1

Question 1, Class Capture

Class capture can be used to consider student answers for the product of two complex numbers.

Note 2

Question 2, Quick Poll

Use a Quick Poll to test comprehension, and then use the Review Workspace to analyze and discuss students' responses.

Note 3

Question 4, Class Capture

Ask students to position z on Page 2.1 to correspond with their descriptions. Use Class Captures to compare student responses.