Consistent Hashing with Bounded Loads

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Abstract

Designing algorithms for balanced allocation of clients to servers in dynamic settings is a challenging problem for a variety of reasons. In such dynamic systems, both servers and clients may be added and/or removed from the system periodically, and the main objectives of allocation algorithms are as follows: the uniformity of the allocation, and the number of moves after adding or removing a server or a client. More specifically, while one main goal is to achieve a proper load balancing in the allocation of clients to servers, after adding/removing clients or servers, re-computing allocations should not result in moving too many clients from their current server. The most popular solution for our dynamic settings is Consistent Hashing [KLL+97, SML+03]. However, the load balancing of consistent hashing [KLL+97, SML+03] is no better than a random assignment of clients to servers, so with n of each, we expect many servers to be overloaded with $\Theta(\log n/\log\log n)$ clients. In this paper, with n clients and n servers, we get a guaranteed max-load of 2 while only moving an expected constant number of clients each time a client or server is added or removed. Our general result is described below.

In this paper, we consider the following problem. We take an arbitrary user specified balancing parameter $c=1+\epsilon>1$. With m balls and n bins in the system, we want no load above $\lceil cm/n \rceil$. Meanwhile we want to bound the expected number of balls that have to be moved when a ball or server is added or removed. Our algorithmic starting point is consistent hashing where current balls and bins are hashed to a unit cycle, and a ball is placed in the first bin succeeding it in clock-wise order. In this paper, we suggest to let bins have capacities, and forward the ball to the first non-full bin. For balancing, we use maximum bin capacity $\lceil cm/n \rceil$. Compared with general lower bounds without capacity constraints, we show that when a ball or bin is inserted or deleted, the expected number of balls that have to be moved is increased only by a multiplicative factor $O(\frac{1}{\epsilon^2})$ for $\epsilon \leq 1$ (Theorem 4) and by a factor $1 + O(\frac{\log c}{c})$ for $\epsilon \geq 1$ (Theorem 3). Technically, the latter bound is the most challenging to prove. It implies that we for superconstant c only pay a negligible cost in extra moves. We also get the same bounds for the simpler problem where we instead of a user specified balancing parameter have a fixed bin capacity C for all bins, and define $c=1+\epsilon=Cn/m$.

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¹Throughout this paper, we use balls and clients, and also bins and servers, interchangeably.

1 Introduction

Load balancing in dynamic environments is a central problem in designing several networking systems and web services [SML+03, KLL+97]. We wish to allocate *clients* (also referred to as *balls*) to *servers* (also referred to as *bins*) in such a way that none of the servers gets overloaded. Here, the *load* of a server is the number of clients allocated to it. We want a hashing-style solution where we given the ID of a client can efficiently find its server. Both clients and servers may be added or removed periodically, and with such changes, we do not want to move too many clients. Thus, while the dynamic allocation algorithm has to always ensure a proper load balancing, it should aim to minimize the number of clients moved after each change to the system. Note that such allocation problems become even more challenging when we face hard constraints in the capacity of each server, that is, each server has a *capacity* and the load may not exceed this capacity. Typically, we want capacities close to the average loads. There is a vast literature on solutions in the much simpler case where the set of servers is fixed and only the client set is updated, but here we will only discuss solutions that are relevant in the fully-dynamic case where both clients and servers can be added and removed.

The classic solution in the scenario where both clients and servers can be added and removed is Consistent Hashing [SML+03, KLL+97] where the current clients are assigned in a random way to the current servers. While consistent hashing schemes minimize the expected number of movements, they may result in hugely overloaded servers, and they do not allow for explicit capacity constraints on the servers. The basic point is that the load balancing of consistent hashing [KLL+97, SML+03] is no better than a random assignment of clients to servers. With m clients and n servers, we may expect a good load balancing if $m/n = \Omega(\log n)$, but the balance is lost with smaller loads, e.g., with m = n, we expect many servers to be overloaded with $\Theta(\log n/\log\log n)$ clients. Using power-of-two-choices [ABKU99] one might hope to get a max-load of $\Theta(\log\log n)$, but in this paper, with n clients and n servers, we get a guaranteed max-load of 2, while only moving an expected constant number of clients each time a client or server is added or removed. Our general result is described below.

In this paper, we present an algorithm that works with arbitrary capacity constraints on the servers. For the purpose of load balancing, we can specify a balancing parameter $c=1+\epsilon$, guaranteeing that the maximum load is at most $\lceil cm/n \rceil$. While maintaining this hard balancing constraint, we limit the expected number of clients to be moved when clients or servers are inserted or removed.

Even without capacity constraints, the obvious general lower bounds for moves are as follows. When a client is added or removed, at least we have to move that client. When a server is added on removed, at least we have to move the clients belonging to it. On the average, we therefore have to move least $\frac{m}{n}$ clients when a server is added or removed.

With our solution, while guaranteeing a balance $c=1+\epsilon \le 2$, when a client is added or removed, the expected number of clients moved is $O(\frac{1}{\epsilon^2})$. When a server is added or removed, the expected number of clients moved is $O(\frac{m}{\epsilon^2 n})$ (Theorem 4). These numbers are only a factor $O(\frac{1}{\epsilon^2})$ worse than the general lower bounds without capacity constrains. For balancing parameter $c \ge 2$, our expected number of moves is increased by a factor $1+O(\frac{\log c}{c})$ over the lower bounds (Theorem 3). The bound for $c \ge 2$ is the most challenging to prove. It implies that we for superconstant c only expect to pay a negligible cost in extra moves.

Compared with previous work, our result is interesting from several perspectives. First of all, it is the first hashing scheme that for any average load can guarantee a constant ratio between maximum and average load while in expectation only moving an asymptotically optimal number of clients when clients or servers are added or removed. Since the maximum load is at least 1, the above statement assumes that the average

load is at least $\Omega(1)$.

From a more practical perspective, our algorithm provides a simple knob, the balancing parameter $c=1+\epsilon$, which captures the tradeoff between satisfying a certain capacity constraint and stability upon changes in the system. As a result, it gives a more direct control to the system designer in meeting explicit capacity constraints.

We get the same bounds for the simpler problem where we instead of a user specified balancing parameter have a fixed bin capacity C for all bins and define $c = 1 + \epsilon = Cn/m$.

Applications Consistent hashing has found numerous applications [ÖV11, GF04] and early work in this area [KLL⁺97, SMK⁺01, SML⁺03] has been cited for more than ten thousand times. To highlight the wide variety of areas in which similar allocation problems might arise, we just mention a few more important references: content-addressable networks [RFH⁺01], peer-to-peer systems and their associated multicast applications [RD01, CDKR02]. Our algorithm is very similar to consistent hashing, and should work for most of the same applications, bounding the loads whenever this is desired.

1.1 Consistent hashing

The standard solution to our fully-dynamic allocation problem is consistent hashing [SML⁺03, KLL⁺97].

Simple consistent hashing. In the simplest version of consistent hashing, we hash the active balls and bins onto a unit circle. Assuming no collisions, a ball is placed in the bin succeeding it in the clockwise order around the circle. One of the nice features of consistent hashing is that it is history-independent, that is, we only need to know the IDs of the balls and the bins and the hash functions, to compute the distribution of balls in bins. If a bin is closed, we just move its balls to the succeeding bin. Similarly, when we open a new bin, we only have to consider the balls from the succeeding bin to see which ones belong in the new bin

With m balls, n bins, and a fully random hash function h, each bin is expected to have m/n balls. This is also the number of balls we expect to move when a bin is opened or closed.

One problem with simple consistent hashing as described above is that the maximum load is likely to be $\Theta(\log n)$ times bigger than the average. This has to do with a big variation in the coverage of the bin. We say that bin b covers the interval of the cycle from the preceding bin b' to b because all balls hashing to this interval land in b. When n bins a placed randomly on the unit cycle, we expect many to cover an interval of size $\Theta(\frac{\log n}{n})$, and we expect $\Theta(\frac{m \log n}{n})$ balls to land in these bins. The maximum load is thus expected to be a factor $\Theta(\log n)$ above the average.

A related issue is that the expected number of balls landing in the same bin as any given ball is almost twice the average. More precisely, consider a particular ball x. Its expected distance to the neighboring bin on either side is exactly 1/(n+1), so the expected size of the interval between these two neighbors is 2/(n+1). All balls landing in this interval will end in the same bin as x; namely the bin b succeeding x. Therefore we expect $2(m-1)/(n+1) \approx 2m/n$ other balls to land with x in b. Thus each ball is expected to land in a bin with load almost twice the average. If the load determines how efficiently a server can serve a client, the expected performance is then only half what it should be.

In [KLL⁺97] they addressed the above issue using virtual bins as described below.

Consistent hashing with virtual bins and a uniform covers. To get a more uniform bin cover, [KLL+97] suggests the use of virtual bins. The virtual bin trick is that the ball contents of $d = O(\log n)$ bins is united

in a single super-bin. The d bins making up a super bin are called virtual bins. We have only n' = n/d super bins and these super bins represent the servers. A super bin covers the intervals covered by its d virtual bins. The point is that for any constant $\epsilon > 0$, if we pick a large enough $d = O(\log n)$, then with high probability, each super bin covers a fraction $(1 \pm \epsilon)/n'$ of the unit cycle.

We note that many other methods have been proposed to maintain such a uniform bin cover as bins are added and removed (see, e.g., [BSS00, GH05, Man04, KM05, KR06]). Different implementations have different advantages depending on the computational model, but for the purpose of our discussion below, it does not matter much which of these methods is used.

With a uniform bin cover, balls distribute almost uniformly between bins. However, with n balls and n bins, we still expect many bins with $\Theta((\log n)/(\log\log n))$ balls even though the average is 1. On the positive side, in the heavily loaded case when m/n is large, e.g., $m/n = \omega(\log n)$, all loads are $(1 \pm \epsilon)m/n$ w.h.p. However, in this paper, we want a good load balancing for all possible load levels.

1.2 Our solution: Respecting bin capacities via forwarding.

In this paper, we consider the case where bins have hard *capacities* that may not be exceeded by the loads. This could both be a direct result of a resource constraint, but it could also be based on a desire for a more strict load balancing, guaranteeing for some balancing parameter c > 1, that no bin has more than $\lceil cm/n \rceil$ balls.

A ball insertion only system Our starting point is simple consistent hashing without virtual bins. To deal with bin capacities, we suggest that if a ball lands in a full bin, that is, the bin succeeding it on the circle is full, then we forward it clockwise around the circle until it finds a bin that is not full, and this ball can be placed in this bin. For now, we only consider insertions of balls with a fixed set of bins. The forwarding is the basic idea behind *linear probing*. We note that for the placement of a ball q, there are two different bins of relevance: the bin b that q hash to in the sense that b is the bin succeeding the hash location of q in the clockwise order, and the the bin b' that q is located in, which is the bin that q got placed in. The location of a ball depends on the order in which the balls are added, e.g., the first ball added will always be placed in the bin it hashes to. Regardless of the insertion order, the forwarding from full bins maintains the following invariant:

Invariant 1. If a ball hashes to bin b and is located in bin $b' \neq b$, then all bins from b till the bin preceding b' are full.

As for linear probing we note that the invariant implies that the full bins are independent of the order in which the balls are inserted. More precisely, we have

Lemma 2. A bin b is full if and only if there is an interval of consecutive bins $B = b_1, ..., b_k$ ending in $b = b_k$ such that the total number of balls hashing to these bins is at least as big as their total capacity.

Proof. If b is full, we take $B = b_1, ..., b_k$ to be the maximum interval of full bins, that is, the bin b' preceding b_1 is not full. By Invariant 1, this means that no balls hashing to or before b' can end in B, so B must be filled with balls hashing to B. In the other direction, the result is trivial if all balls hashing to B end in B, since there are enough balls to fill all bins. However, if a ball hashing to B ends up after b_k , then b_k is full by Invariant 1.

Locating a ball in the bins To search a ball q in the current bins, we first search the bin b it hashes to. If q is not found and if b is full, we move to the next bin b' and recurse until either we find q or we reach a bin b' without q that is not full. In the latter case, by Invariant 1, we conclude that q is not in the system. If q is to be inserted, it should be inserted in b'.

Supporting other updates Now, we want a system that handles not only insertion of balls, but also deletion of balls plus addition and removal of bins, and we need to maintain Invariant 1. If a bin is removed, we just reinsert its balls. If a ball q is deleted from a bin b, we have to check if a ball q' in a succeeding bin can fill the hole because it hashes before b, but then recursively, we have to fill the whole left by q'. Likewise, when a bin b is inserted, we have to check if balls in succeeding bins can fill it, just like we filled the hole resulting from a deletion.

History independence If we want our solution to be history-independent, we can use the idea from [BG07], that the balls are placed following the order of their IDs, ignoring the order in which they are actually introduced. We talk about IDs as being ordered from lower to higher.

In practice, we work with hash functions with a limited range $[r] = \{0,, r-1\}$. Mapping this range to a circle, position 0 succeeds r-1. We assume independent hash functions for the balls and the bins. For our bounds we only assume that $r \geq n$ and that both the ball hash function and the bin hash function are 5-independent (simple tabulation [PT12] will also work even though it is only 3-independent). With limited-range hash functions, we may have collisions. To get a complete cyclic order, we do the following tie breaking: if two balls or two bins hash to the same location, then the one with the lower ID precedes the one with the higher ID. Moreover, if a ball and a bin hash to the same location, the ball precedes the bin. This implies that the bins hashing to a given position x will always be filled bottom-up.

The above describes exactly how given balls are to be distributed among given bins with given capacities using a given hash function. If a ball or a bin is inserted or deleted, or a bin capacity is changed, then balls have to be moved to match the resulting new distribution. If we do not care about history-independence, then the tie breaking among balls and among bins can be done arbitrarily, e.g., following the historical order in which they were inserted. The analysis below still holds (it corresponds to the special case of using increasing IDs).

While the idea of consistent hashing with forwarding to meet capacity constraints seems pretty obvious, it appears not to have been considered before. Our main theoretical contribution is the analysis described below.

1.3 Load balancing

We will now focus on load balancing. For a given load balancing parameter $c=1+\epsilon>1$, we want to guarantee that no bin has more than $\lceil cm/n \rceil$ balls. One possibility would be to just say that all bins had capacity $\lceil cm/n \rceil$, but then adding a single ball could force us to increase the capacity of all bins, completely changing the configuration. As a result, we need to be careful about enforcing the above capacity constraints across all bins. In particular, to minimize the number of capacity changes when balls are inserted or deleted, we aim for a total bin capacity of $\lceil cm \rceil$, letting the lowest $\lceil cm \rceil - n \lfloor cm/n \rfloor$ bins have capacity $\lceil cm/n \rceil$ while the rest have capacity $\lfloor cm/n \rfloor$. We refer to the former bins as big bins and the latter bins as small bins, though the difference is only 1. Moreover, as an exception to the above rule, we will never let the capacity drop below 1, that is, if cm < n, then all bins have capacity 1. By adding or removing a ball, we affect the capacity of at most $\lceil c \rceil$ bins.

Subject to the capacity constraints, our main focus in this paper is the number of balls that have to be moved when a ball or bin is inserted or deleted. Mathematically, the most interesting case is when $c = \omega(1)$. We note that inserting a ball results in up to $\lceil c \rceil$ bins increasing their capacity. Nevertheless, besides placing the new ball, we will prove that the expected number of ball moves is $O((\log c)/c) = o(1)$. The general result is

Theorem 3. For a given load balancing parameter $c \ge 2$, the expected number of bins visited in a search is $1 + O((\log c)/c)$. When a ball is inserted or deleted, the expected number of other balls that have to be moved between bins is $O((\log c)/c)$. When a bin is inserted or deleted, besides moving O(m/n) expected balls hashing directly to the bin, we expect to move $O((m/n)(\log c)/c)$ other balls.

For the insertion and deletion of bins, Theorem 3 implies that the expected number of moves is O(m/n). The reason that we distinguish the balls hashing directly to the bin is to allow a direct comparison with standard consistent hashing without capacity constraints [KLL⁺97]. For standard consistent hashing, the balls affected by the insertion or deletion of a bin are exactly the balls hashing to it. We expect O(m/n) such balls (previously, we have said that exactly m/n hash to any given bin, but that was assuming ideal fully random hash functions). With our capacity constraints, we only expect to move $O((m/n)(\log c)/c)$ other balls. The price we pay for guaranteeing a maximum load of $\lceil cm/n \rceil$ is thus only a multiplicative factor $1 + O((\log c)/c) = 1 + o(1)$ in the expected number of ball moves.

For $c \in (1, 2]$, we parameterize by $\varepsilon = c - 1 > 0$.

Theorem 4. For a given load balancing parameter $c = 1 + \varepsilon \in (1, 2]$, the expected number of bins visited in a search is $O(1/\varepsilon^2)$. When a ball is inserted or deleted, the expected number of other balls that have to be moved between bins is $O(1/\varepsilon^2)$. When a bin is inserted or deleted the expected number of balls that have to be moved is $O(m/(n\varepsilon^2))$.

The bounds of Theorem 4 are similar to those obtained in [PT12] for linear probing. The challenge here is to deal with the fact that bins are randomly placed, as opposed to linear probing where every hash location has a bin of size 1. Nevertheless we will be able to reuse some of the key lemmas from [PT12]. The proof of Theorem 3 is far more challenging, and the main focus of this paper.

Remark 5. The bounds from Theorems 3 and 4 also hold in the simpler case where all bins have a fixed capacity C and we define $c = 1 + \epsilon = Cn/m$. We note that our updates change the value of m and m, hence of c = Cn/m. For the bounds to apply, we always use the smaller value of c in connection with each update. Thus, for the bounds on the moves in connection with a ball insertion or bin removal, we use the value of c before the update. For the bounds on the moves in connection with a ball deletion or bin addition, we use the value of c after the update.

Computational model. In this paper, we do not pay much attention to concrete computational aspects. Consistent hashing is a simple versatile scheme that has been used in many different systems offering different computational options [ÖV11, GF04]. The basic thing we need is a mechanism that given a point on the unit cycle can find the next bin in the clockwise order. The system Chord [SML+03] offers a nice way to do this in a certain distributed environment. They let each bin maintain pointers to $O(\log n)$ bins, and then the bin containing a point is found exchanging $O(\log n)$ messages. This is the message cost for finding the server of a client with simple consistent hashing.

Here we also let each bin maintain the succeeding and preceding bins in the clockwise order so that the current bins form a a doubly-linked cycle. We can then get from a bin to its successor exchanging O(1) messages. Thus, to find the bin of a ball as in Theorem 4, we first find the bin that ball hash to using $O(\log n)$ messages as in Chord [SML⁺03]. Next we expect to search $O(1/\epsilon^2)$ successive bins using $O(1/\epsilon^2)$ messages. This extra work is negligible if $\varepsilon = \omega(1/\sqrt{\log n})$. This concrete analysis is for a Chord-like implementation of our system.

In connection with system updates, that is, addition and removal of balls and bins, we discuss in Section 5, how to efficiently compute the balls to be moved.

Once again, the versatility of consistent hashing with its more than ten thousand citations is very much due to it being a very simple algorithm that can be implemented in different environments with different performance criteria. Our consistent hashing with load balancing via forwarding is almost as simple, and offers an attractive alternative whenever load balancing is important. Because we are not concerned with the concrete implementation, our focus is on analyzing the fundamental combinatorial properties like loads, the number of bins searched, and the number of balls that are moved when the system is updated.

1.4 Power of choice as an alternative?

We have proposed using forwarding in the style of linear probing to bound the loads in consistent hashing. Below we discuss the possibility of instead using the power of choice [ABKU99, PR01, Mit01, BCSV00, TW14, Vöc03]. In its most basic form [ABKU99] we have a fixed set of n bins. Balls are added, one by one. Each ball is given d uniformly random bins to choose from, and picks the least loaded, breaking ties arbitrarily. With m balls, w.h.p., we end up with a maximum load of $m/n + \frac{\ln \ln n}{\ln d} + \Theta(1)$ [BCSV00]. An interesting twist suggested by Vöcking is that if several bins have the same smallest load, we always pick the left-most [Vöc03]. Surprisingly, this reduces the max load to $m/n + \Theta(1 + \frac{\ln \ln n}{d})$ [Vöc03]. We note that to get a constant ratio between maximum and average load when the average load is constant, we do need a super constant number of choices, e.g., $d = \Omega(\log \log n)$ with left-most choice.

We now argue that with our current understanding, it is not clear how well power of choice will work in the context of consistent hashing. The above mentioned bounds are proved in the ideal case where we pick uniformly between the bins. Consider now first the case of simple consistent hashing where both balls and bins are placed at random on a unit circle, and a ball goes to the succeeding bin in clockwise order. This case is studied in [BCM03, BCM04], where it was proved that if m=n, then the maximum load is $O(\log\log n)$. However, with a concrete example, [Wie07] show that we cannot in general hope to get maxload $m/n + O(\log\log n)$ when $m\gg n\log\log n$. This is again for the case of simple consistent hashing where bins are just placed randomly on the circle. However, using, e.g., virtual bins, we know that that we can obtain a more uniform bin cover such that each bin represents a fraction $(1\pm\epsilon)/n$ of the unit cycle and where a ball lands in a bin with this probability. With $\varepsilon=1/2$, the main result from [Wie07] implies that using the power of d choices, w.h.p., we get a maximum load of $m/n + O(\frac{\log\log n}{\log d})$.

We still have to consider what happens if balls are deleted and if bins are added or removed. The results from [CFM $^+$ 98] indicate that to delete a ball, it may suffice to just remove it without moving any other balls. However, if a bin b is removed, we have to move all its balls. If b covers a single interval, as in consistent hashing, the balls in b will at first be transferred to the succeeding bin b' which would typically get overloaded by a factor 2. We then have to consider the other choices for the balls, and see if that helps. With the fragmented cover of virtual bins, the problem might be less severe. With virtual bins, the contents of each of super bin b's virtual bins would be transferred to the succeeding virtual bin, but these successors typically belong to different super bins, leading to a better spread of the balls from the removed super bin b. Nevertheless, we may still need to consider the other choices for some of the balls from b.

It may very well be that one can get power of choice to work with consistent hashing and get bounded loads even in a fully-dynamic environment where both balls and bins get added and removed. In order to

prove any meaningful bounds, we would need to specify which choices should be reconsidered in connection with system updates, most notably, the addition and removal of bins. The hope would be that we would only need to reconsider the choices of balls where the current choice is directly associated with the bin added or removed.

Dreaming further, for our load balancing with parameter c, we specified capacities $\lfloor cm/n \rfloor$ or $\lceil cm/n \rceil$ for each bin in such a way that the total capacity was $\lceil cm \rceil$ and such that we only changed few bin capacities in connection with each update. We then used forwarding in the style of linear probing to respect these capacities. Obviously, one could try to adapt many other hash table schemes to respect these capacities. Most notably, we could hope to adapt Cuckoo hashing [PR04] where we only have two choices for each ball, just like in power of two choices discussed above. Both are designed for a fixed set of bins. With power of two choices, we do not change the choices of previously placed balls, but with Cuckoo hashing, we do reconsider previous choices. Cuckoo hashing [PR04] is designed for capacity 1. If we want to place a ball in a full bin, we can just ask the ball in the bin to go for its other choice, possibly pushing out a ball in that location, thus resulting in a chain of moves. This chain is unique with capacity 1, and successful if it doesn't loop. With larger capacities, when we place a ball, we need to chose which ball to push out, leading to an exponentially growing tree of choices. At the end, we only need to do the chain of moves of one branch. This issue was considered in [Pan05] for capacity 2. Here we would like a protocol working with larger capacities and in the fully dynamic setting where both balls and bins can be added and removed. This may indeed be possible, and would be interesting.

Stepping back, the first contribution of this paper is to raise the issue of getting consistent hashing with strict load balancing even in the lightly loaded case (with average load $\omega(\log n)$, w.h.p., we get good load balancing using consistent hashing with virtual bins [KLL⁺97]). Our main result is that the problem can be solved using the forwarding idea of linear probing, the main technical contribution being in the analysis.

It may very well be that the load balancing issue can also be solved with some variant of multiple choices. This would indeed be interesting, but it would not dominate our solution based on forwarding. As an example where we have an advantage, take the Chord-style model [SML+03] considered earlier. In this model, we pay $\Theta(\log n)$ messages to find the bin containing a hash value representing a choice. With multiple choices, if d choices are considered to find a ball, then d becomes a multiplicative factor on the message cost. For contrast, with our forwarding, we only have to find one bin containing a hash value. Subsequently, we only consider consecutive bins, paying only $\Theta(1)$ messages to get from one bin to its successor. With constant balance parameter c, we only expect to consider a constant number of successors, so the expected overhead is only an additive constant. Moreover, the expected number of moves in connection with updates was only a constant factor bigger than the general lower bounds. Thus we expect our solution to remain relevant regardless of future developments with multiple choices.

2 High Level Analysis

To analyze the expected number of moves, we shall use a general probabilistic understanding of configurations encountered. By a configuration, we refer to the situation after the balls via forwarding have settled into the right bins. Suppose we have m balls and n bins inserted. We refer to them as active. We will also talk about passive balls and bins that are not currently in the system, yet which have hash values that will be used if they get inserted. For some $\bar{c} \geq c$, the total capacity will be exactly $\bar{c}m$. Since no bin has capacity below 1, we always have $\bar{c}m/n \geq 1$. In our analysis, we will only assume that each bin has a capacity between $\bar{c}m/(2n)$ and $2\bar{c}m/n$. We note that this is always satisfied when bin capacities differ at most by 1.

We are going to prove:

Theorem 6. Consider a configuration with m active balls and n active bins and total capacity $\bar{c}m$ for some $\bar{c} \geq 2$. Suppose, moreover, that each bin has capacity between $\bar{c}m/(2n)$ and $2\bar{c}m/n$. Then

- (a) Starting from the hash location of a given passive ball or active or passive bin, the expected number of consecutive filled bins is $O(1/\bar{c})$.
- (b) If we start from a given active bin of capacity at least 2, the expected number of consecutive filled bins is $O((\log \bar{c})/\bar{c}^2)$.
- (c) The expected number of balls hashing directly to any given active bin is O(m/n). The expected number of balls forwarded into the bin is $O((m/n)((\log \bar{c})/\bar{c}^2))$. Finally, if a bin is not active, and its active successor q is given an extra capacity of one, then the expected number of full bins starting from q is $O((\log \bar{c})/\bar{c}^2)$.

The above statements are satisfied if the balls and bins are hashed independently, each using 5-independent hash functions or simple tabulation hashing. The statement of (c) may seem a bit cryptic, and will make more sense in the context of the analysis it is used in below.

It should be noted that the worst-case for our bounds is when the capacities are 1 and 2. This also explains why (a) would not work for an active ball since an active ball by itself could fill a bin. However, when a ball is inserted, it goes in the nearest non-full bin in the configuration before its insertion while it is still passive.

Corollary 7. With balancing parameter $c = 1 + \varepsilon \in (1, 2]$, the expected number of bins visited in a search is $1 + O((\log c)/c)$ if $c \ge 2$, and $O(1/\varepsilon^2)$ if $c \le 2$.

Proof. The proof doesn't belong here, but it has a cute element to it. If the ball q searched is not in the system, then the search is to only up to and including the first non-full bin. However, if q is in the system, the latest it can be placed is if was added last, which corresponds is the first non-full bin if in the system without q. This means that in both cases, the expected search is bounded as one plus the number of consecutive filled bins starting from an passive ball as in Theorem 6 (a).

2.1 Bounding the expected number of moves

We are now going to prove Theorem 3 using Theorem 6 with different values of $\bar{c} \geq c$, where c is the fixed parameter chosen to control the bin capacity. The bounds only get better with larger \bar{c} , and we know that before and after every update we have total capacity $\lceil cm/n \rceil = \bar{c}m/n$ so $\bar{c} \geq c$. For the sake of the analysis, we will be careful about the order in which we insert balls, increase capacities, etc. The configurations before and after an update are completely fixed, however, and we are really only counting the net moves of balls, which is independent of the concrete implementation.

Plain insertion Consider an insertion of a ball q hashing to some bin i. For now, we ignore that some capacities have to be increased. First we place q in i. If bin i was not already full, we are done. Otherwise, we take the highest ball q' from i and move it to the next bin i'. It would be q' = q if q is higher than all the balls already in bin i. If i' was already full, we repeat the process, terminating when we get to the first bin that was not already full. The important thing to note is that we only move one ball per full bin encountered, and that we stop when we meet a bin that is not full before the insertion. By Theorem 6 (a) applied to the configuration before the insertion, the expected number of moves, excluding the initial placement, is $O(1/\bar{c}) = O(1/c)$.

Plain deletion The deletion of a ball q is the inverse of inserting it. This follows easily from our history-independent configurations where the balls, bins, and capacities uniquely determine the location of balls in bins. It follows that the expected number of moves, excluding the final removal, is $1 + O(1/\bar{c})$ but now \bar{c} is measured right after the deletion where we have only m-1 balls so $\bar{c} = \lceil cm/n \rceil m/(m-1) > c$.

Full deletion To complete a deletion, we have to do $c\pm 1$ capacity decreases, all together bringing the the capacity down to $\lceil c(m-1) \rceil$. Doing one capacity decrease at the time, we know that every configuration encountered has total capacity $C \ge \lceil c(m-1) \rceil$, and hence $\bar{c} = Cm/n \ge c$. We also note that the bins picked for capacity decreases are chosen independently of the random choices made by the hash functions.

When we decrease the capacity of a bin i, if the bin now has one ball more than its capacity, we perform exactly the same process as when we hashed a ball to a full bin, forwarding the highest ball until we reach a bin that was not already full. The number of moves is bounded by the number of consecutive full bins starting from i in the configuration before the capacity of bin i was decreased.

A crucial observation is that since the lowest possible capacity is 1, bin i had capacity at least two before the decrease, so by Theorem 6 (b) applied to the configuration before the capacity decrease, the expected number of moves is $O((\log \bar{c})/\bar{c}^2) = O((\log c)/c^2)$. We are doing at most $\lceil c \rceil$ such capacity decreases, so the total expected number of moves, including the removal of q itself, is bounded by

$$1 + O(1/c) + (1+c)O((\log c)/c^2) = 1 + O((\log c)/c).$$

Full insertion A full insertion is the inverse of a full deletion of a ball q, so the expected number of moves, including the addition of q itself is $1 + O((\log c)/c)$.

Closing a bin When closing a bin q, we are going to lose its capacity C, which is cm/n rounded up or down. Our first action is to increase by one the capacities of C other bins. Doing the increases first, we make sure that the total capacity before every increase is always at least cm. An increase is the inverse of the decrease that we studied under full deletions, so the expected number of moves for each increase is bounded by $O((\log c)/c^2)$. The expected total number of moves resulting from all C bin increases is thus

$$C \cdot O((\log c)/c^2) = O((m/n)O((\log c)/c).$$

We are now ready to start closing q, having made sure that the total capacity remains above cm. The immediate consequence is that the balls in q are transferred to its successor q'. The balls in q are either hashed directly to q, or they are forwarded from the predecessor of q. By Theorem 6 (c), we expect to have O(m/n) balls hashed directly to q and $O(m/n(\log c)/c)$ forwarded to q from its predecessor.

Instead of transferring the balls from q directly to q', we first take them out of the system, and close q. We now have a valid configuration with $m' \leq m$ balls, and the total capacity is at least $cm \geq cm'$. We now reinsert the balls from q one by one in q'. We pay an extra move every time a ball is forwarded because a bin overflow. As discussed under plain insertions, when inserting a ball in q', the number of such overflow moves is exactly the number d of full consecutive bins starting from q'. The number d can only increase as balls are inserted. Before inserting the last ball, the number d is exactly the same as it would be if all balls had been inserted but bin q' had an extra unit of capacity to hold the last ball. We note here that in a correct configuration, it would probably be a different lower ball that would occupy the extra capacity, but this does not affect which bins are full. Thus, by Theorem 6 (c), the expected largest number d of full bins encountered is at most $O((\log c)/c^2)$. We also know, by our hard capacity constraint, that the maximal

number of balls transferred from q is $C \leq \lceil cm/n \rceil$. We conclude that the expected number of forwarding moves is bounded by $\lceil cm/n \rceil O((\log c)/c^2) = O((m/n)(\log c)/c)$.

Summing up, we have proved that when closing q, besides the transfer to q' of O(m/n) expected balls hashing directly to q, we have $O((m/n)(\log c)/c)$ ball moves. Here n is the number of bins before the closing.

Opening a bin is the reverse of closing it, so the number of moves is the same with n counting the number of bins after the opening. This completes the proof of Theorem 3 assuming the correctness of Theorem 6.

2.2 Small capacities

In the same way that we proved Theorem 3 assuming Theorem 6, we can prove Theorem 4 using the following theorem.

Theorem 8. Consider a configuration with n active balls and n active bins and total capacity $(1 + \bar{\varepsilon})m$ for some $\bar{\varepsilon} \in (0,1]$. Suppose, moreover, that each bin has capacity between $(1 + \bar{\varepsilon})n/(2n)$ and $2(1 + \bar{\varepsilon})m/n$. Then starting from the hash location of a given passive ball or active or passive bin, the expected number of consecutive filled bins is $O(1/\bar{\varepsilon}^2)$.

Here we assume that balls and bins are hashed independently, each using 5-independent hash functions or simple tabulation hashing.

As in the analysis for Theorem 3, we will be operating with an $\bar{\varepsilon} \geq \varepsilon$ such that the total capacity is exactly $(1+\bar{\varepsilon})m$. Applying Theorem 8 will then yield a bound of $O(1/\bar{\varepsilon}^2) = O(1/\varepsilon^2)$. However, it could be that while $\varepsilon \leq 1$, we end up with $\bar{\varepsilon} > 1$. In this case, we will instead apply Theorem 6, and get a bound of $O((\log(1+\bar{\varepsilon}))/(1+\bar{\varepsilon})) = O(1) = O(1/\varepsilon^2)$.

We note that Theorem 8 is much simpler than Theorem 6. The point is that Theorem 6 is used to prove a loss by a factor 1 + o(1) relative to standard consistent hashing without capacities. For Theorem 4, the loss factor is $O(1/\varepsilon^2)$, which is much less delicate to deal with, e.g., for the number of balls in a bin, instead of analyzing the expected number, which is O(m/n), we can just use the hard capacity, which is at most $2(1 + \bar{\varepsilon})m/n = O(m/n)$.

2.3 Basic probability bounds

We will now briefly review the probability bounds we will use, and what demands they put on the hash functions used. The general scenario is that we have some bounded random variables $X_1, \ldots, X_n = O(1)$ and $X = \sum_{i=1}^n X_i$. Let $\mu = \mathbf{E}[X]$. Assuming that the X_i are 4-independent, we have the fourth moment bound

$$\Pr[|X - \mu| \ge x] = O((\mu + \mu^2)/x^4).$$
 (1)

Deriving (1) is standard (see, e.g., [PPR09]). We will typically have the variable X_i denoting that a ball (or a bin) hash to a certain interval, which may be defined based on the hash of a certain query ball q. If the hash function is 5-independent, the X_i are 4-independent.

The fourth moment bound is very useful when $\mu \geq 1$. For smaller μ , we are typically looking for upper bounds x on X, and there we have much better bounds in the combinatorial case where X_i indicates if ball (or bin) i lands in some interval. Suppose we know that the X_i are a-independent for some $a \leq x$. The probability that a given balls land in the interval is p^a , so the expected number of such a-sets is $\binom{m}{a}p^a$. If we get x or more balls in the interval, then this includes at least $\binom{x}{a}$ such a-sets in the interval. Thus, by Markov's inequality, with independence $a \leq x$,

$$\Pr[X \ge x] = (\mu^{\underline{a}}/a!)/(x^{\underline{a}}/a!) = O((\mu/x)^a) \text{ for } a = O(1)$$
 (2)

where x^a is defined to be $x(x-1)\cdots(x-a+1)$. We shall only use (2) with $a\leq 3$. One advantage to this is that our results will hold, not only with 5-independent hashing, but also with simple tabulation hashing. The point is that in [PT12] it is shown that while simple tabulation is only 3-independent, it does satisfy the fourth moment bound (1) even with a fixed hashing of a given query ball. According to the experiments [PT12], simple tabulation hashing is an order of magnitude faster than 5-independent hashing implemented via a degree-6 polynomial.

3 Analysing expectancies with large capacities

In this section, we are now going to prove Theorem 6. Cheating a bit, we will assume $\bar{c} \geq 64$, and instead prove Theorem 8 for any $\bar{\varepsilon} = O(1)$, noting that for $\bar{\varepsilon}, \bar{c} = \Theta(1)$, the bounds in both theorems are just constant.

Also, to increase readability, since the parameter $c \le \bar{c}$ does not appear in the analysis, we will just write c instead of \bar{c} below. First we focus on proving Theorem 6 (a) and (b) as restated in the following lemma.

Lemma 9. Starting from the hash location of a given q, which is either a passive ball or an active or passive bin, the expected number of consecutive filled bins is O(1/c). If q is a given bin with capacity at least 2, the expected number of consecutive filled bins is $O((\log c)/c^2)$.

Proof. We bound the expected number d of consecutive filled bins around h(q). These should include q if q is a bin, and the bin q would hash to if q is a passive ball. If this bin is not full, then d=0 and we are done. Otherwise, let $I=(a,b]\ni h(q)$ be the interval covered by these full bins, that is, a is the location of non-filled bin preceding the first filled bin, and b is the location of the last filled bin. In our analysis, we assume that h(q) is fixed before the hashing of any other balls and bins.

We will study the event that $t^- \le d < t^+$ and $\ell^- < |I| \le \ell^+$. First we note that $d \ge t^-$ and $|I| \le \ell^+$ imply the event:

 $A(t^-,\ell^+)$: Enough balls to fill t^- bins hash to $(h(q)-\ell^+,h(q)+\ell^+)$.

Also $d < t^+$ implies that at most $t^+ - 2$ bins land in $I \setminus \{b\} = (a,b)$. We are discounting position b, because we could have additional bins hashing to b that are not full. Note that we could have h(q) = b. No matter how $I = (a,b] \ni h(q)$ is placed, we have either $(a,b) \supseteq [h(q) - \lceil \ell^-/2 \rceil, h(q))$, or $(a,b) \supseteq [h(q), h(q) + \lceil \ell^-/2 \rceil)$. Thus $d < t^+$ and $|I| > \ell^-$ imply the event:

 $B(t^+,\ell^-)$: Either at most t^+-2 bins hash to $[h(q)-\lceil\ell^-/2\rceil,h(q))$ or at most t^+-2 bins hash to $[h(q),h(q)+\lceil\ell^-/2\rceil)$.

Since balls and bins hash independently, after h(q) has been fixed, the events $A(t^-,\ell^+)$ and $B(t^+,\ell^-)$ are independent, so

$$\Pr[t^- \le d < t^+ \land \ell^- < |I| \le \ell^+] \le \Pr[A(t^-, \ell^+)] \Pr[B(t^+, \ell^-)].$$

For $i=0,1,\ldots$, let $t=2^i,\,t^-=t$, and $t^+=2t$. Moreover, define $\ell(t)=\lfloor 8tr/n\rfloor$. Recall here that $r\geq n$

is the range we hash balls and bins to to. We will bound $\Pr[t \le d < 2t]$ as follows.

$$\Pr[t \le d < 2t] \le \Pr[t \le d < 2t \land |I| \le \ell(t)]$$

$$+ \sum_{j=0}^{\lceil \log_2 c \rceil} \Pr[t \le d < 2t \land 2^j \ell(t) < |I| \le 2^{j+1} \ell(t)]$$

$$+ \Pr[t \le d < 2t \land c\ell(t) < |I|]$$

$$\le \Pr[A(t, \ell(t))]$$

$$+ \sum_{j=0}^{\lceil \log_2 c \rceil} \Pr[A(t, 2^{j+1} \ell(t))] \Pr[B(2t, 2^j \ell(t))]$$

$$+ \Pr[B(2t, c\ell(t))].$$
(3)

The main motivation for the definition of $\ell(t)$ is that with $\ell^- = s\ell(t)$, $s \ge 1$, we can get a good fourth moment bound on $\Pr[B(2t, \ell^-)]$.

We consider the case where among all bins different from q (which may be a ball or a bin), at most $x=t^+-2=2(t-1)$ hash to $\lceil h(q)-\lceil \ell^-/2\rceil, h(q) \rceil$. We note that there are at least n-1 bins that are different from q. We will pay a factor 2 in probability to cover the equivalent case where they hash to $\lceil h(q), h(q)+\lceil \ell^-/2\rceil \rceil$. The expected number of bins different from q hashing to $\lceil h(q)-\lceil \ell^-/2\rceil, h(q) \rceil$ is $p \geq (n-1)\lceil \ell^-/2\rceil/r$, and we want this to be at least $p \geq 2$. This is indeed the case with $p \geq 2$ we have

$$\mu \ge (n-1)\lceil \ell^-/2 \rceil/r \ge ns\ell(t)/(4r) \ge st \ge 1.$$

Applying the fourth moment bound (1), we now get

$$\Pr[B(2t, s\ell(t))] = O((\mu + \mu^2)/(\mu - 2(t-1))^4) = O(1/\mu^2) = (1/(st)^2). \tag{4}$$

Next, we will develop different bounds for $\Pr[A(t,s\ell(t))]$ that are all useful in different contexts. The interval $(h(b)-s\ell(t),h(b)+s\ell(t))$ has length $2s\ell(t)-1$, so the expected number of active balls hashing to it is $\mu < 2s\ell(t)n/r = O(stm/n)$. However, to satisfy $\Pr[A(t,s\ell(t))]$, we need enough balls to fill t bins in that interval. Their total capacity is $x \geq tcm/(2n)$. Suppose we also know that $x \geq a$ and that the balls hash a-independently. Since we only assume 3-independence, $a \leq 3$. However, if q is a ball, with h(q) fixed, we are left with 2-independence. Now, from (2), we get the bound

$$\Pr[A(t, s\ell(t))] = O((\mu/x)^a) = O((s/c)^a).$$
(5)

Thus, from (3), we get

$$\Pr[t \le d < 2t] \le \Pr[A(t, \ell(t))] + \left(\sum_{j=0}^{\lceil \log_2 c \rceil} \Pr[A(t, 2^{j+1}\ell(t))] \Pr[B(2t, 2^{j}\ell(t))]\right) + \Pr[B(2t, c\ell(t))] \\
= O(1/c^a + \left(\sum_{j=0}^{\lceil \log_2 c \rceil} (2^{j+1}/c)^a (1/(2^j t)^2)\right) + 1/(ct)^2 \\
= \begin{cases}
O(1/c) & \text{if } a = 1 \\
O(1/c^2 + (\log c)/(ct)^2) & \text{if } a = 2 \\
O(1/c^3 + 1/(ct)^2) & \text{if } a = 3
\end{cases}$$
(6)

When we start from the hash of an passive ball or bin, or an active bin whose capacity might be only one, we can use a=1 for t=1 bin and a=2 for $t\in[2,c]$. We now get

$$\begin{split} \mathbf{E}[d \cdot [d \leq c]] &\leq \sum_{t = 2^{j}, j = 0, \dots, \lfloor \log_{2} c \rfloor} 2t \Pr[t \leq d < 2t] \\ &= O(1/c) + \sum_{t = 2^{j}, j = 1, \dots, \lfloor \log_{2} c \rfloor} t O(1/c^{2} + (\log c)/(ct)^{2}) \\ &= O(1/c). \end{split}$$

Above, for a Boolean expression B, we have [B] = 1 if B is true, and [B] = 0 if B is false.

In the case where we start from a bin q with capacity at least 2, we can use a=2 for t=1 bin. After the fixing of h(q), the balls are still hashed 3-independently and the capacity is at least 3 with $t \ge 2$ bins, so for $t \in [2, c]$, we can use a=3. Thus, when q is a bin with capacity at least 2, we get

$$\begin{aligned} \mathbf{E}[d \cdot [d \le c]] &= \sum_{t=2^j, j=0, \dots, \lfloor \log_2 c \rfloor} 2t \Pr[t \le d < 2t] \\ &= O((\log c)/c^2) + \sum_{t=2^j, j=1, \dots, \lfloor \log_2 c \rfloor} t O(1/c^3 + 1/(ct)^2) \\ &= O((\log c)/c^2). \end{aligned}$$

For $t \geq c$, we are instead going to use a fourth moment bound on $\Pr[A(t,\ell(t))]$. To satisfy $A(t,\ell(t))$, we need $x \geq tcm/(2n)$ active balls to hash to $(h(q) - \ell(t), h(q) + \ell(t))$ but the expectation is only $\mu = (2\ell(t) - 1)n/r \leq 2\lfloor 8tr/n \rfloor n/r \leq 16tm/n$. Since $c \geq 64$, we have $x \geq 2\mu$. Moreover, $\mu \geq 16$ since $cm/n \geq 1$ and $t \geq c$. Hence, by (1), we get

$$\Pr[A(t,\ell(t))] = O((\mu + \mu^2)/(t - \mu)^4) = O(\mu^2/t^4) = O((tm/n)^2/((tcm/n)^4)) = O(1/(tc)^2)$$
 (7)

For $t \ge c$, this replaces the $1/c^a$ term in (6), so we get

$$\begin{split} \Pr[t \leq d < 2t] \leq & \Pr[A(t,\ell(t))] + \sum_{j=0}^{\lceil \log_2 c \rceil} \Pr[A(t,2^{j+1}\ell(t))] \Pr[B(2t,2^{j}\ell(t))] + \Pr[B(2t,c\ell(t))] \\ = & O(1/(ct)^2) + \sum_{j=0}^{\lceil \log_2 c \rceil} (2^{j+1}/c)^a (1/(2^{j}t)^2 + 1/(ct)^2) \\ = & \begin{cases} O(\log c)/(ct)^2) & \text{if } a = 2 \\ O(1/(ct)^2) & \text{if } a = 3 \end{cases} \end{split}$$

In the general case where we start from any passive ball, or active or passive bin, we have a=2 for $t\geq 2$, so we get

$$\begin{aligned} \mathbf{E}[d \cdot [d \ge c]] &= \sum_{t = 2^j c, j = 0, \dots, \infty} 2t \Pr[t \le d < 2t] \\ &= \sum_{t = 2^j c, j = 0, \dots, \infty} 2t O((\log c)/(ct)^2) = O(((\log c)/c^2), \end{aligned}$$

and then

$$\mathbf{E}[d] = \mathbf{E}[d \cdot [d < c]] + \mathbf{E}[d \cdot [d \ge c]] = O(1/c) + O(((\log c)/c^2)) = O(1/c).$$

This completes the proof of the first statement of the theorem.

For the case where q is a bin with capacity at least 2, we have a=3 for $t\geq 2$, and therefore

$$\mathbf{E}[d \cdot [d \ge c]] = \sum_{t=2^{j}c, \ j=0,\dots,\infty} 2t \Pr[t \le d < 2t]$$

$$= O(\sum_{t=2^{j}c, \ j=0,\dots,\infty} t \ O(1/(ct)^{2})$$

$$= O(1/c^{2}).$$

Therefore, when q is a bin with capacity at least 2,

$$\mathbf{E}[d] = \mathbf{E}[d \cdot [d < c]] + \mathbf{E}[d \cdot [d \ge c]] = O(((\log c)/c^2) + O(1/c^2) = O(((\log c)/c^2).$$

This completes the proof of the theorem.

We will now prove Theorem 6 (c) restated below.

Lemma 10. The expected number of balls hashing directly to any given active bin q is O(m/n). The expected number of balls forwarded into q from its predecessor q^- is $O(m/n(\log c)/c^2)$. Finally, if a bin is not active, and its active successor q^+ is given an extra capacity of one, then the expected number of of full bins starting from q^+ is $O((\log c)/c^2)$.

Proof. For the first statement, we note that the expected number of balls hashing to each location h(q)-i is n/r for any $0 \le i \le r$. These are not added to q if some bin hash to [h(q)-i,h(q)), which is an independent event because balls and bins hash independently. The expected number of bins hashing to [h(q)-i,h(q)) is $\mu=i(n-1)/r$. For $i \ge r/(n-1)$, we have $\mu \ge 1$, and then, by (1), the probability of getting no bins in [h(q)-i,h(q)) is $O((\mu+\mu^2)/(\mu-0)^4)=O(1/\mu^2)=O((r/(ni))^2)$. The expected number of balls hashing directly to q is thus bounded by

$$n/r \cdot \left(\lfloor r/(n-1) \rfloor + \sum_{i=\lfloor r/(n-1) \rfloor + 1}^{\infty} (r/(ni))^2 \right) = O(m/n).$$

We also have to consider the probability that the preceding bin q^- forwarded balls to q. For this to happen, we would need q^- to be filled even if we increased its capacity by 1, and then q^- would have capacity at least 2. This is bounded by the probability of having an interval $I \ni h(q) - 1$ with $d \ge 1$ consecutive full bins including one with capacity at least 2. This is what we analyzed in the proof of Lemma 9, so we get $\Pr[d \ge 1] \le \mathbf{E}[d] = O((\log c/c^2))$. By the capacity constraint, the maximal number of balls that can be forwarded to and end in q is 2cm/n, so the expected number is

$$O((\log c/c^2)2cm/n = O((m/n)(\log c)/c).$$

Next we ask for the expected number d of full bins starting from the active successor q^+ of a given passive bin q, when q^+ is given an extra capacity of one. Again this implies that q^+ has capacity at least 2, and then the analysis from the proof of Lemma 9 implies that $\mathbf{E}[d] = O((\log c)/c^2)$.

4 Small capacities

In this section we will prove Theorem 8 restated below with ε instead of $\bar{\varepsilon}$ and allowing any positive $\varepsilon = O(1)$ instead of just $\varepsilon \in (0, 1]$.

Lemma 11. Consider a configuration with n active balls and active n bins and total capacity $(1 + \varepsilon)m$ for some $\varepsilon = O(1)$. Suppose, moreover, that each bin has capacity between $(1 + \varepsilon)n/(2n)$ and $2(1 + \varepsilon)m/n$. Then starting from the hash location of a given passive ball or active or passive bin, the expected number of consecutive filled bins is $O(1/\varepsilon^2)$.

Here we assume that balls and bins are hashed independently, each using 5-independent hash functions or simple tabulation hashing.

We shall sometimes use a weighted count for number Y_I of bins hashing to an interval I, where the weight of a bin is its capacity divided by the average capacity $(1+\varepsilon)m/n$. These weights are between 1/2 and 2. The total capacity of the bins hashing into I is precisely $Y_I(1+\varepsilon)m/n$.

Below we choose δ such that $(1+\delta)/(1-\delta)=(1+\varepsilon)$. For $\varepsilon \leq 1$, we have $\delta \geq \varepsilon/3$, but we also have for any $\varepsilon = O(1)$ that $\delta = \Omega(\varepsilon)$. For $d \geq 6$, our goal is to show that the probability of getting d consecutive full bins is

$$O(1/d^2\delta^4) = O(1/d^2\varepsilon^4). \tag{8}$$

It will then follow that the expected number of full bins is $O(1/\varepsilon^2)$, as required for Lemma 11.

Before doing the probabilistic analysis, we consider the combinatorial consequences of having d consecutive full bins.

Lemma 12. Let p be the hash location of an open or closed bin or an inactive ball from which the number of successive full bins is $d \ge 2$. If we do not have d balls hashing directly to p, there there is an interval I containing p of length at least $\ell = |dr/(2n)|$, where either

- (i) the number balls X_I landing in I is $X_I \ge (1+\delta)m|I|/r$, or
- (ii) the weighted number of bins Y_I in I is $Y_I \leq (1 \delta)n|I|/r$.

Proof. Let I = (a, b] be the longest interval covered by consecutive full bins around p. More precisely, a is the hash location of the non-full bin preceding the first full bin in the interval. We note here that the first full bins hash strictly after a because bins hashing to the same location always get filled bottom-up. Since the preceding bin at a is not full, bins in I can only be filled with balls hashing to I.

Since we do not have d bins hashing to p, and we had d full bins starting from p, we must have p < b, so $p \in (a,b)$. Also, we have at least d full bins in I, so I must contain at least $d(1+\varepsilon)n/(2n)$ balls. Suppose $|I| \le \ell$. We can then expand I in either end to an interval $I^+ \supseteq I$ of length $\ell = \lfloor dr/(2n) \rfloor$, and then $X_{I^+} \ge X_I \ge (1+\varepsilon)dn/(2n) \ge (1+\varepsilon)n|I^+|/r > (1+\delta)n|I^+|/r$, so (i) is satisfied for I^+ . Thus we may assume that $|I| \ge \ell + 1$.

We now look at the interval $I^- = (a, b)$ which contains p and is of length at least ℓ . All bins in I^- are full. Even though our last full bin hashed to b, we have to exclude b because we might also have non-full bins hashing to b.

Since all bins hashing to I^- are filled with balls hashing to I^- , we have $X_{I^-} \ge (1 + \varepsilon)Y_{I^-}m/n$. We now reach a contradiction if (ii) and (i) are false for I^- , for then

$$X_{I^{-}} < (1+\delta)m|I^{-}|/r = (1+\varepsilon)(1-\delta)m|I^{-}|/r < (1+\varepsilon)Y_{I^{-}}m/n.$$

We note that applying (2) with a=2, we get that the probability of d bins hashing to p is $O(1/d^2)=O(1/d^2\delta^4)$. Hence it suffices to bound the probability of (i) and (ii). We will do this using a technical result from [PT12].

To state the result from [PT12], we first reconsider the standard fourth moment bound (1). We are hashing m balls into [r]. Let $\alpha = n/r$ be the density of the balls in r. Let I be an interval that may depend on the hash location of the inactive query ball, and let X_I be the number of active balls hashing to I. Then $\mathbf{E}[X_I] = \alpha |I|$, and hence we can state (1) as

$$\Pr[|X_I - \alpha|I|| \ge \delta \alpha |I|] = O\left(\frac{\alpha |I| + (\alpha |I|)^2}{(\delta \alpha |I|)^4}\right)$$
(9)

Now [PT12] considered the general event $\mathcal{D}_{\ell,\delta,p}$ that, for a given $p \in [r]$ that may depend on the hash value of an inactive query ball, there exists an interval I containing p of length at least ℓ such that the number of keys X_I in I deviates from the mean $\alpha|I|$ by at least $\delta\alpha|I|$. As a perfect generalization of (1), [PT12] proved

$$\Pr[\mathcal{D}_{\ell,\delta,p}] = O\left(\frac{\alpha\ell + (\alpha\ell)^2}{(\delta\alpha\ell)^4}\right)$$
(10)

The proof of (10) in [PT12] only assumes (9), even though (9) only considers a single interval whereas (10) covers all intervals around a given point of some minimal length.

The bound (10) immediately covers the probability that there exists an interval $I\ni p$ of length at least ℓ which satisfies (i). As stated earlier, we may assume that $d\ge 6$, and since $r\ge n$, we get that $\ell=\lfloor dr/(2n)\rfloor\ge dr/(3n)$. Moreover, the total capacity is $(1+\varepsilon)m\le O(m)$, and the minimum bin capacity is 1, so $n\le O(m)$, and therefore $\alpha\ell\ge dn/(3n)=\Omega(1)$. Thus the probability of (i) is bounded by

$$O\left((\alpha\ell + (\alpha\ell)^2)/(\delta\alpha\ell)^4\right) = O(1/(\delta^4d^2)) = O(1/(\delta^4d^2)).$$

The analysis above could be made tighter if $m \gg n$, but then the error probability would just be dominated by that for (ii) on bins studied in the next part.

We now want to limit the probability of getting too few bins, that is, for some $I \ni p$ containing p of length at least $\ell \ge dr/(3n)$,

(ii) the weighted number of bins in I is $Y_I \leq (1 - \delta)n|I|/r$

It makes no difference to (10) from [PT12] that we apply it to bins instead of balls. We note the bin counts are weighed with weights below 2. This is not an issue because weights where considered in [PT12], and the equations (1) and (10) both hold for weights bounded by a constant. Another technical issue is if we in Lemma 11 start from an active query bin at p. Then this bin is always included in the interval $I \ni p$. It should only have been included with probability |I|/r. A simple solution is to only count the start bib with this probability, yielding a slightly smaller value $Y_I' \le Y_I$. We then instead bound the probability that $Y_I' \le (1 - \delta)n|I|/r$, which is implied by (ii). Now the bin density is exactly $\alpha = m/r$ and $\alpha \ell \ge (n/r)dr/(3n) \ge d/3 \ge 2$, so the probability of (ii) is bounded by

$$O\left((\alpha\ell + (\alpha\ell)^2)/(\delta\alpha\ell)^4\right) = O(d^2/(\delta d)^4) = O(1/(\delta^4 d^2).$$

We have thus shown that $O(1/(\delta^4 d^2))$ bounds the probability of both (i) and (ii) in Lemma 12, hence the probability of getting $d \geq 1/\delta^2$ successive full bins. It follows that the expected number of successive bins is bounded by

$$1/\delta^2 + \sum_{d=\lceil 1/\delta^2 \rceil}^{\infty} O(1/(\delta^4 d^2)) = O(1/\delta^2) = O(1/\varepsilon^2)$$

This completes the proof of Lemma 11, and hence of Theorem 8.

5 Computing moves when the system is updated

In this paper, we have not been concerned about computing the moves that have to be done. In many applications this is a non-issue since we can afford to recompute the configuration from scratch before and after each update. These are also the applications where history independence matters. Our more efficient computation of moves is based on a global simulation of the system in RAM. Our implementation is tuned to yield the best theoretical bounds.

We will now first describe an efficient implementation when balls are just inserted in historical order, much like in the standard implementation of linear probing. We will describe how to compute the moves associated with an update in expected time proportional to the bounds we gave above for the expected number of moves.

Given a ball q, we want to find the bin it hashes to in expected constant time. This is the bin succeeding q clockwise when bins and balls are hashed to the unit cycle. For that we use an array B of size $t = \Theta(n)$. Entry i is associated with the interval $I_i = [i/t, (i+1)/t)$. In B[i], we store the head of a list with the bins hashing to I_i in sorted order. The list has expected constant size. When a ball q comes, we compute the interval $I_i \ni h(q)$, and check B[i] for its succeeding bin. If L[i] has no bin hashing after q, we go to L_{i+1} . The expected time is O(1). As bins are inserted and deleted, we use standard doubling/halving techniques in the back ground to make sure that L always have $\Theta(n)$ entries.

For each bin, we store the balls landing in it in a hash table. Recall here that when a ball is inserted, we first find the bin it hashes to, but if it is full, we have to place it in the first non-full succeeding bin. Such insertions are trivially implemented in time proportional to the number of bins considered, which is exactly what we bounded when we considered the number of moves.

We now turn to deletions. A deletion is essentially like a deletion in linear probing. When we take out a ball q from a bin b, we try to refill the hole by looking at succeeding bins b' for a ball q' that hash to or before b. We then move q' to b, and recurse to fill the hole left by q' in b'. The last hole created is the bin where q would land if q was inserted last in the current configuration. We are willing to pay time proportional to number of bins from the one q hashed to, and to the one it would land in if inserted last.

To support efficient deletions, we let each bin b have a forwarding count for the number of balls that hashed preceding b but landed in a bin succeeding b. Also, we divide the balls landing in b according to which bins they originally hashed to. More precisely, we maintain a doubly-linked list with the bins that balls landing in b hashed to, sorted in the same order as the bins appear on the cycle. With each of these preceding bins b^- , we maintain a list with the balls that hashed to b^- and landed in b. The total space for these lists is proportional to the number of balls landing in b, so the total space remains linear. We assume that we can access the sorted bin list from each end.

Now, if a deletion of a ball q creates a hole in b, then this hole can be filled if and only if the forwarding counter of b is non-zero. If so, we consider the succeeding bins b' one by one. In b', we go to the beginning of the sorted bin list to find the first bin b^- that a ball landing in b' hashed to. If b^- equals or precedes b, we can use any ball q' from the list of b^- in b' and fill the hole in b. Checking the bin b' and possibly moving a ball takes constant time. If a hole is created in b' and b' has non-zero forwarding counter, we recurse.

Another issue is that we have is to locate a ball to be deleted in one of the above lists. For that we employ an independent hash table for the current balls point to their location. The hash table could be implemented with linear probing which works in expected constant time using the same 5-independent or simple tabulation hash function that we used to map balls to the unit cycle [PPR09, PT12]. Below it is assumed that we always update the hash table with the location of the balls as we move them around. An alternative solution would be that we searched the ball starting from the bin it hashed to, and then only used

a local hash table for each bin.

When a ball q is inserted, we have to update the above information. If it hashes to a bin b and lands in a bin b', then we have to increment the forwarding counter for all bins starting from b and going to the bin preceding b'—if b = b', no forwarding counter is incremented. Now inside b', we have to go backwards in the list of bins that balls landing in b' hashed to, searching for b, inserting b if necessary. Next we add a to the list of a in a. The bins considered inside a is a subset of the bins we passed from a to a to the bound is not increased.

When bins are inserted and deleted, we implement the effect using insertion and deletion of the affected balls as described above.

The most interesting challenge is that when a ball is inserted or deleted, we have to change the capacity of c bins. This becomes hard when $c = \omega(1)$ where we only have O(1) expected time available.

As the system is described above, we let the lowest $\lceil cm \rceil - n \lfloor cm/n \rfloor$ bins have large capacity $\lceil cm/n \rceil$ while the rest have small capacity $\lfloor cm/n \rfloor$. However, we only have to guarantee that no bin has capacity above $\lceil cm/n \rceil$.

We now relax things a bit. With no history independence, we just maintain the list of current bins in the order they were inserted (if one is deleted and reinserted, we count it as new). The large bins form a prefix of this list. We also relax the requirement on the number of large bins to be $\lceil cm \rceil - n \lfloor cm/n \rfloor \pm c$. This means that when a ball is inserted or deleted, it doesn't have to be exactly c bins whose capacity we change. Next we partition the list of current bins into groups of length at most c with the property that the combined length of any two consecutive groups is at least c. This partition is easily maintained in constant time per bin update, including pointers so that we can from each bin can find the head of the group it belongs to.

The changes to bin capacities are now done for one or two groups at the time, telling only the group heads what the capacity is. Thus, to check if a bin is full, we have to ask the group head about the capacity.

Now, when we change the capacity of a group, we find out if the change in capacity means that some bins need changes to their information. A bin requiring action is called critical. More precisely, a large bin is critical if it is full. A small bin is critical if it is full and its forwarding count is non-zero. This implies that it would be full if it became large. By Lemma 9, a bin is critical with probability $O((\log c)/c^2)$, so we only expect $O((\log c)/c) = o(1)$ critical bins in a group. To identify critical bins efficiently, we divide groups into subgroups of size at most \sqrt{c} , using the same algorithm as we used for dividing into groups. For each group and for each subgroup, we count the number of critical bins. Now a group only has a critical bin with probability $O((\log c)/c)$, which is our expected cost to check if there are critical bins. If so, we check which subgroups have a positive count of critical bins. If a subgroup has critical bins, we find them by checking all the bins in the subgroup. Altogether, we pay $O(\sqrt{c})$ time per critical bin, so the expected time to identify the critical bins is $O(\sqrt{c}(\log c)/c^2)) = o(1)$.

We next consider an implementation with history independence. For history independence, we first do a random permutation of the bin and ball IDs. We can use the classic $\pi(x) = ax + b \mod p$ where p is a random prime and a and b are uniformly random in \mathbb{Z}_p . These permuted IDs are still history independent. The point now is that if we have a set X of $\Theta(k)$ permuted IDs, then we can maintain order in this set by bucketing based on $\lfloor \pi(x)/k \rfloor$. The buckets are then in relative sorted order, and we can easily maintain order within each bucket since each ID is expected to end in bucket with O(1) other IDs. As usual, we can use doubling/halving if the set X grows or decreases by a constant factor. Now it is easy to maintain an ordered list of permuted bin IDs where a prefix of these are large bins, just like in our original description. Also, for each bin b, we maintain the list of balls that hash to it, ordered based on the permuted IDs. We note that the balls in this list are extracted based on the hashing to the unit cycle which is independent of the permutation of the ball IDs. The balls hashing to b are placed in b and succeeding bins based on the ordering

of this list, that is, first we fill b with the balls with the lowest permuted IDs. In particular, the list of balls hashing to b but landing in bin b' is just a segment of the sorted list of balls hashing to b.

6 Simulation Results

To validate the consistency property of our hashing scheme which is theoretically analysed in Theorems 3, and 4, we present the following empirical results. We generated thousands of instances, and tracked the number of ball movements in each operation, and the distribution of bin sizes. We picked the number of bins, n, the average balls per bin ratio $r = \frac{m}{n}$, and ϵ as follows:

- $n \in \{10, 20, 40, 70, 100, 150, 200, 300, 450, 600, 800, 1000, 2000\}.$
- $r = \frac{m}{n} \in \{0.5, 0.8, 1, 1.2, 1.5, 2, 3, 5, 10\}$
- $\epsilon \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.2, 1.5, 1.8, 2, 2.3, 2.5, 2.8, 3\}$

Figures 1 and 2 show the distribution of bin loads for our algorithm and Consistent Hashing algorithm respectively. Figures 3 and 4 depict the average number of movements in our algorithm for each value of ϵ . We start with Figure 1 that shows the distribution of bin loads. The three plots represent three values of $\epsilon \in \{0.1, 0.3, 0.9\}$. We expect the load of each bin to be at most $\lceil (1+\epsilon)m/n \rceil$. To unify the results of various simulations with different average loads (m/n), we divide the loads of bins by m/n, and sort the normalized bin loads in a decreasing order (breaking ties arbitrarily). The y coordinate is the normalized load, and the x coordinate shows the fractional rank of the bin in this order. For instance if a bin's load is greater than 35% of other bins, its x coordinate will be 35%. As we expect, no bin has normalized load more than $1+\epsilon$. A significant fraction of bins have normalized load $1+\epsilon$, and the rest have normalized loads distributed smoothly in range $[0, 1+\epsilon]$. The smaller the ϵ is, the more we expect bins to have normalized loads equal to $1+\epsilon$, and consequently having a more uniform load distribution.

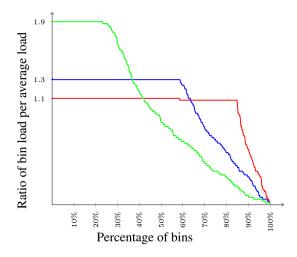


Figure 1: Bin loads divided by average load m/n for our algorithm.

As shown above, the maximum load never exceeds $1 + \epsilon$ times the average load in our algorithm. However, you can see below that there is no constant upper bound on maximum load for Consistent

Hashing algorithm. We simulate consistent hashing with n balls and n bins for three different values of $n \in \{200, 1000, 8000\}$. As expected, the maximum load grows with n, it is expected to be around $\log(n)/\log\log(n)$. The percentage axis is rescaled to highlight the more interesting bin sizes.

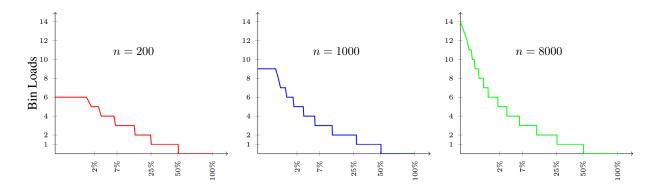


Figure 2: Bin loads divided by average load for Consistent Hashing algorithm.

Figure 3 depicts the number of movements per ball operation. Theorems 3, and 4 suggests that the expected number of movements per ball operation is $O(\log(c)/c)$, and $O(1/\epsilon^2)$ respectively where c is $1+\epsilon$. The solid red curve in Figure 3 depict the average normalized ball movements in all simulations for each value of ϵ . The bars show the standard deviation of these normalized movements. The dashed black line is the upper bound on these numbers of movements predicted by Theorems 3 and 4 with the following formula:

$$f(\epsilon) = \begin{cases} 2/\epsilon^2 & \text{if } \epsilon < 1\\ 1 + \log(1 + \epsilon)/(1 + \epsilon) & \text{if } \epsilon \ge 1 \end{cases}$$

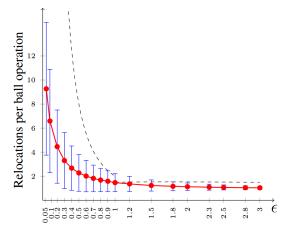


Figure 3: Number of movements per ball operation for different values of ϵ .

Our results predict that unlike the ball operations, the average movements per bin operation (insertion or deletion of a bin) is proportional to average density of bins $r = \frac{m}{n}$. Therefore we designate Figures 4 and 3

to bin and ball operations respectively. We start by elaborating on Figure 4. Theorems 3 and 4 suggest that the average number of movements per bin operation is $O(r\log(c)/c)$, and $O(r/\epsilon^2)$ respectively where c is $1+\epsilon$. To unify the results of all our simulations, we normalize the number of movements in bin operations with dividing them by r. The solid red curve in Figure 4 depict the average normalized bin movements in all simulations for each value of ϵ . The bars show the standard deviation of these normalized movements. The dashed black line is the upper bound function, $f(\epsilon)$ (defined above), on these numbers of movements predicted by Theorems 3 and 4. Similarly Figure 3 shows the relation between the number of movements and ϵ for ball insertions and deletions. These are the actual number of movements and are not normalized by the average density r.

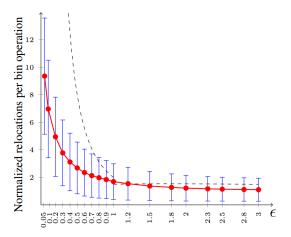


Figure 4: Normalized number of movements per bin operation for different values of ϵ .

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