

**Linkage principle, Multi-dimensional Signals  
and  
Blind Auctions**

**Thierry Foucault      Stefano Lovo<sup>1</sup>**

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<sup>1</sup>Both authors are from the HEC School of Management, 1 rue de la Libération, 78351 Jouy en Josas, France. Foucault can be reached at: Tel: (33) 1 39 67 94 11; e-mail: foucault@hec.fr. Lovo can be reached at: Tel: (33) 1 39 67 74 62; e-mail: lovo@hec.fr.

## **Abstract**

We compare the seller's expected revenue in a second price sealed bid auction for a single object in which bidders receive multidimensional signals. Bidders' valuations for the object depend on their signals and a signal observed privately by the seller. We show in various examples that the seller can be better off not revealing publicly his signal. Hence the linkage principle does not necessarily hold when bidders receive multidimensional signals.

**Keywords:** Auction Theory, Linkage Principle, Multidimensional Signals, Blind Auctions.

**JEL Classification :** D44

# 1 Introduction

The “linkage principle” is a very important result of auction theory. An implication of the linkage principle is that a seller maximizes his expected revenue by publicly revealing his information regarding the value of the object for the bidders. This result has been established by Milgrom and Weber (1982).<sup>1</sup> It holds true in single-unit auctions, with symmetric bidders, when bidders’ valuations are affiliated. Recently Perry and Reny (1999) have shown, by way of a counter-example, that this is not necessarily true in multi-unit auctions. Building on this counter-example, Krishna (2002) shows that the release of public information may generate a decrease in seller’s revenue when bidders are asymmetric in a single-unit auction.<sup>2</sup> In this paper, we complement these negative findings regarding the linkage principle. We show that the linkage principle can fail in a single unit auction when bidders are symmetric but receive multidimensional signals.

We establish this result by considering “blind auctions”. A blind auction is an auction in which bidders do not know the object which is auctioned. However they know that the object is of one among  $m$  possible types and they know their private valuation for each type. The seller knows the type of the object but does not observe bidders’ valuations. A bidder’s value for the object is therefore determined by  $m$  (real value) signals (his private valuation for each type) and the signal possessed by the seller. The main difference with the standard independent private value framework is the fact that each bidder has multiple signals.

We compare the seller’s expected revenue when he runs a blind auction and when he *publicly* releases perfect information on the type of the object. In either format, the seller uses a sealed bid second price auction. We show that when there are two bidders the seller’s expected revenue is *always* strictly larger in the blind auction, i.e. when he does *not* publicly reveal the type of the object. Moreover, using various examples, we show that the seller’s expected revenue can also be strictly larger in the blind auction when the number of bidders is larger than 2. As we do not depart from the independent private value framework, the revenue equivalence theorem

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<sup>1</sup>See Krishna (2002), Chapter 7 for an exposition of the linkage principle.

<sup>2</sup>A related paper is Bergemann and Pesendorfer (2003). They consider an optimal auction design problem in which the seller can also choose the information structure of each bidder. They show that optimal information structures are “partitional” and thus leaves some uncertainty on their valuations to the bidders. The linkage principle does not apply in their setting because information is not released publicly and is unknown to the seller (i.e. remains private to each bidder). Finally, they do not consider the case in which bidders receive multi-dimensional signals

applies and our results extends to the first price auction and to other commonly used auction mechanisms.

Interestingly blind auctions are actually used in financial markets. Increasingly portfolio managers auction packages of trades to brokers. Brokers announce a trading fee per share at which they commit to execute each stock in the package at its closing price on the day preceding the auction.<sup>3</sup> The broker announcing the lowest fee wins the auction and will be in charge of executing **all** the trades in the package (see Kavajecz and Keim (2002) for a detailed description of these auctions). The winning broker makes a profit if he achieves lower transaction costs than his fee.<sup>4</sup> This can be done by clearing many of the orders in the package against his own book of pending transactions.<sup>5</sup> A key feature of these auctions is that portfolio managers *do not reveal* the identity of the securities in the package and the number of shares to buy or sell for each security. Hence brokers (i.e., bidders) face some uncertainty regarding the “object” (a package of trades) which is auctioned.

The paper is organized as follows. Section 2 presents the model. Section 3 shows that it can be optimal for a seller to conduct a blind auction. Section 4 concludes.

## 2 Model

A seller wishes to sell one object. There are  $n$  risk neutral bidders. The seller and each bidder receive real value private signals regarding the value of the object. The seller receives signal  $S \in \{1, 2, \dots, m\}$ , with  $\Pr(S = j) = \pi_j$  and  $m \geq 2$ . Bidder  $i$  receives signal  $X_i = (X_i^1, X_i^2, \dots, X_i^m)$ , where the components  $X_i^j$  are identically and independently distributed over  $[0, 1]$ . We denote by  $F(\cdot)$  the cumulative distribution of signal  $X_i^j$  and by  $f(\cdot)$  its density. Bidders' signals  $X_i$  are i.i.d. and they are independently distributed from  $S$ . Each bidder and the seller observes his own signal, but not the signals received by the other agents. We denote by  $V_i = u(X_i, S)$  the

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<sup>3</sup>Hence each broker commits to sell (buy) the stocks in the package at a fixed price plus a markup (resp. minus a discount) which is the trading fee per share.

<sup>4</sup>The transaction cost on a specific trade is the difference between the average price at which the broker executes the trade and the guaranteed price.

<sup>5</sup>Since each broker has a different book of pending transactions, it is not unreasonable to view the brokers as having different private values for each possible composition of the package.

random variable that represents bidder  $i$ 's valuation for the object. We assume that

$$u(x_i, s) = \sum_{j=1}^m x_i^j \mathbf{1}_{\{s=j\}}, \quad (1)$$

where  $(x_i, s)$  is a given realization of  $(X_i, S)$  and  $\mathbf{1}_{\{s=j\}}$  takes the value 1 if  $s = j$  and 0 otherwise. Notice that a bidder's valuation does not depend on the other bidders' valuations and that  $u(., .)$  is identical for all bidders. Hence bidders have private valuations and are symmetric. Finally we note that all signals are affiliated.<sup>6</sup>

One interpretation for traders' valuation is as follows. The object sold by the seller has  $m$  possible types (for instance, the object has  $m$  possible different colors). Type  $j$  is obtained with probability  $\pi_j$ . Bidders have private valuations for the object that depend on its type.

The seller organizes a second price sealed bid auction. Each bidder submits a sealed bid. The bidder who submits the highest bid receives the object and he pays the seller the second highest bid. The seller can decide (before receiving his signal) to *publicly* and *perfectly* reveal his signal or not before running the auction. We assume that the seller can credibly commit to a disclosure policy.<sup>7</sup> If the seller does not disclose his signal, we say he runs a *blind auction* since in this case bidders do not know exactly the type of the object.

Suppose that the seller receives signal  $s$ , and bidder  $i$  receives signal  $x_i = \{x_i^1, x_i^2, \dots, x_i^m\}$ . If the seller reveals his signal, then bidder  $i$ 's valuation will be  $v_i = u(x_i, s) = x_i^s$ , whereas if the seller does not reveal his signal, then bidder  $i$ 's expected valuation will be  $w_i = E[u(x_i, S)] = \sum_{j=1}^m \pi_j x_i^j$ . Observe that bidders' valuations are independent whether the type of the object is known or not. Thus, as is well-known, each bidder has a weakly dominant bidding strategy which consists in bidding his valuation for the object.

In the rest of the paper, we study the seller's expected revenue in each possible format for the auction.

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<sup>6</sup>For all  $i, j$ , signals  $S$ , and  $X_i^j$ , are affiliated as independence is a special case of affiliation. Moreover, let  $h$  be the joint distribution of  $V_i$  and  $S$ . Then we have  $h(v_i, s) = f(v_i)\pi_s$ . Thus, for any  $v, v', s$  and  $s'$  it results  $h(v, s)h(v', s') = h(v, s')h(v', s)$ . Therefore  $V_i$  and  $S$  are affiliated.

<sup>7</sup>This is not important when traders' valuations are given by Eq.(1). In this case, once the seller opts for disclosure, it is weakly dominant for him to truthfully reveal his information since signals  $X_i^j$  are i.i.d for a given  $i$ .

### 3 Findings

#### 3.1 Two bidders

If the seller reveals his signal  $s$ , then the winner of the object is the bidder with the highest  $x_i^s$  and he pays the second highest valuation among the  $n$  bidders conditional on the type of the object being  $s$ . Let  $X_{(2)}^s$  be the second highest element of  $\{X_1^s, X_2^s, \dots, X_n^s\}$ . By assumption, all the signals  $X_i^j$  are i.i.d. for all  $i, j$ . It follows that the  $X_{(2)}^s$  are i.i.d. for all  $s \in \{1, 2, \dots, m\}$  with cumulative distribution

$$F_{(2)}(z) = F(z)^n + nF(z)^{n-1}(1 - F(z)).$$

Hence the seller's expected revenue ex-ante (before receiving the signal) is

$$\begin{aligned} R_D &= \sum_{j=1}^m \pi_j E[X_{(2)}^j] = E[X_{(2)}^1] = \\ &= \int_0^1 z f_{(2)}(z) dz = 1 - \int_0^1 F_{(2)}(z) dz, \\ &= 1 + \int_0^1 ((n-1)F(z)^n - nF(z)^{n-1}) dz, \end{aligned} \tag{2}$$

where  $f_{(2)}(\cdot) \stackrel{def}{=} F'_{(2)}(\cdot)$ .

If the seller runs a blind auction, then bidder  $i$  bids his *expected valuation* for the object, i.e.,  $w_i = \sum_{j=1}^m \pi_j x_i^j$ . The winner of the object is the bidder with the highest expected valuation for the object and he pays the second highest expected valuation among the  $n$  bidders. Let  $G$  be the cumulative distribution of  $W_i = \sum_{j=1}^m \pi_j X_i^j$ . As the  $W_i$  are i.i.d., the seller's expected revenue with the blind auction is

$$R_{ND} = E[W_{(2)}] = 1 + \int_0^1 ((n-1)G(z)^n - nG(z)^{n-1}) dz. \tag{3}$$

**Proposition 1** : *If  $n = 2$  then the seller's expected revenue is larger in the blind auction, i.e.  $R_D < R_{ND}$ .*

**Proof:** For  $n = 2$ , we have  $R_D < R_{ND}$  iff

$$\int_0^1 F(z)^2 - 2F(z) dz < \int_0^1 G(z)^2 - 2G(z) dz \tag{4}$$

As the random variables  $X_i^j$  are i.i.d. for all  $i$  and  $j$ , we have  $E[W_i] = \sum_{j=1}^m \pi_j E[X_i^j] = E[X_1^1]$ . As  $E[W_i] = E[X_1^1]$ , we have  $\int_0^1 z g(z) dz = \int_0^1 z f(z) dz$  where  $g(\cdot) \stackrel{\text{def}}{=} G'(\cdot)$ . Integrating by parts we deduce that  $\int_0^1 F(z) dz = \int_0^1 G(z) dz$ . Hence Condition (4) rewrites

$$\int_0^1 (F(z)^2 - G(z)^2) dz < 0. \quad (5)$$

Observe that the random variable  $W_i$  is a weighted average of  $m$  i.i.d. random variables with cumulative distribution  $F(\cdot)$ . Thus  $G$  second order stochastically dominates  $F$ .<sup>8</sup> Hence the following inequality holds true:

$$\int_0^x (F(z) - G(z)) dz \geq 0 \quad \forall x \in [0, 1],$$

and it is strict for some  $x \in (0, 1)$  as  $F \neq G$ . Let  $\theta(z) \stackrel{\text{def}}{=} \int_0^z (F(y) - G(y)) dy$ . The function  $\theta(\cdot)$  has the following properties: (i)  $\theta'(z) = F(z) - G(z)$ ; (ii)  $\theta(z) \geq 0$  for all  $z \in [0, 1]$ ; (iii)  $\theta(z) > 0$  for some  $z \in (0, 1)$ ; (iv)  $\theta(0) = 0$  and (v)  $\theta(1) = 0$  since  $\int_0^1 F(y) dy = \int_0^1 G(y) dy$ . Thus,

$$\begin{aligned} \int_0^1 (F(z)^2 - G(z)^2) dz &= \int_0^1 (F(z) + G(z)) \theta'(z) dz \\ &= [(F(z) + G(z)) \theta(z)]_0^1 - \int_0^1 (f(z) + g(z)) \theta(z) dz \\ &= 0 - \int_0^1 (f(z) + g(z)) \theta(z) dz < 0. \blacksquare \end{aligned}$$

### 3.2 More than 2 bidders

The previous proposition shows that when there are 2 bidders, the seller is always strictly better off not disclosing his information on the type of the object. Thus the linkage principle does not necessarily hold true in our framework. Now we show, using 2 numerical examples, that this conclusion also holds true when there are several bidders. In each example we assume that  $m = 2$  and  $\pi_1 = \pi_2 = 1/2$ . Thus  $W_i = \frac{1}{2}(X_i^1 + X_i^2)$ . The cumulative probability distribution of the signals,  $F(\cdot)$ , is different in each example.

**Example 1.** Assume that  $X_i^j$  can take two values, say 0 and 1 with probability  $(1 - p)$  and  $p$ . Consider the case in which the seller discloses his signal. The price paid by the winner of

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<sup>8</sup>For any strictly concave function  $U : \mathbb{R} \rightarrow \mathbb{R}$  and any vector  $\{x_i^1, \dots, x_i^m\} \in [0, 1]^m$  realization of  $X_i$ , we have  $U(\sum_{j=1}^m \pi_j x_i^j) \geq \sum_{j=1}^m \pi_j U(x_i^j)$ , with strict inequality if  $x_i^j$  are not all identical. Thus,  $E[U(\sum_{j=1}^m \pi_j X_i^j)] > E[\sum_{j=1}^m \pi_j U(X_i^j)]$ . That is  $E[U(W_i)] > E[U(X_i^j)]$ . This means that  $G(\cdot)$  second order stochastically dominates  $F(\cdot)$ .

the auction is 1 iff at least 2 buyers bid 1 for the object (in which case, the object is randomly assigned to one of the 2 buyers). This occurs iff at least two among the  $n$  bidders have a valuation equal to 1. Otherwise the price paid by the winner of the auction is zero. Hence the seller's expected revenue is

$$R_D = \Pr(X_{(2)}^s \geq 1) = 1 - (1 - p)^{n-1}(1 + (n - 1)p).$$

Now consider the case in which the seller does not disclose his signal. In this case, the probability distribution of each bidder's valuation ( $W_i$ ) is:

$$\begin{cases} \Pr(W_i = 0) = (1 - p)^2 \\ \Pr(W_i = \frac{1}{2}) = 2p(1 - p) \\ \Pr(W_i = 1) = p^2 \end{cases}$$

We deduce that the cumulative distribution of the second highest valuation among the  $n$  bidders is

$$\begin{cases} \Pr(W_{(2)} \leq 0) = (1 - p)^{2(n-1)}(n + (1 - n)(1 - p)^2), \\ \Pr(W_{(2)} \leq 0.5) = (1 - p^2)^{n-1}(1 + p^2(n - 1)), \\ \Pr(W_{(2)} \leq 1) = 1. \end{cases}$$

The price paid in the auction is equal to the second highest valuation among the  $n$  bidders. Thus, the seller's expected revenue if he does not disclose his information is

$$\begin{aligned} R_{ND} &= 0.5(\Pr(W_{(2)} \leq 0.5) - \Pr(W_{(2)} = 0)) + (1 - \Pr(W_{(2)} \leq 0.5)) = \\ &= 1 + \frac{1}{2}(1 - p)^{2(n-1)}((n - 1)p(p - 2) - 1) - \frac{1}{2}(1 - p^2)^{n-1}(1 + (n - 1)p^2). \end{aligned}$$

The next table indicates for various pairs  $(p, n)$  whether the seller is better off disclosing (the cell contains a "D") or not ("ND").

$p/n$	2	3	4	5	6	7	10	12	15	20	25	30
0.5	ND	D	D	D	D	D	D	D	D	D	D	D
0.4	ND	ND	D	D	D	D	D	D	D	D	D	D
0.3	ND	ND	ND	D	D	D	D	D	D	D	D	D
0.2	ND	ND	ND	ND	ND	D	D	D	D	D	D	D
0.1	ND	ND	ND	ND	ND	ND	ND	ND	D	D	D	D

**Table 1**



**Example 2.** In our second example, the bidders' signals are drawn from a continuous distribution. Specifically, we assume that  $X_i^j$  is continuously distributed on  $[0, 1]$  with a cumulative probability distribution:

$$F(z) = \frac{z + \alpha \log(1+z)}{1 + \alpha \log(1+z)} \quad (6)$$

with  $\alpha \geq 0$ . Notice that  $F(z)$  increases with  $\alpha$ .<sup>9</sup> This means that the larger is  $\alpha$ , the larger is the probability that bidders have small valuations. In this sense, increasing  $\alpha$  is similar to decreasing  $p$  in the previous example. The cumulative probability distribution for a bidder's valuation if the seller does not disclose his information is then:

$$\Pr(W_i < z) = G(z) = \int_0^1 f(y)F(2z - y)dy$$

Using Equations (2) and (3) in the previous subsection, we compute numerically  $R_D$  and  $R_{ND}$  for various pairs  $(\alpha, n)$ . Table 2 indicates for which pairs  $(\alpha, n)$ , the seller is better off disclosing (the cell contains a "D") or not ("ND").

$\alpha \backslash n$	2	3	4	5	6	7	8	9	10
2	ND	ND	D	D	D	D	D	D	D
4	ND	ND	ND	D	D	D	D	D	D
10	ND	ND	ND	D	D	D	D	D	D
40	ND	ND	ND	ND	ND	ND	ND	D	D

**Table 2**

**Summary.** In these 2 examples, the seller is better off with the blind auction if the number of bidders is small and/or the probability of observing  $X_i^j$  close to zero is sufficiently large ( $p$  small/ $\alpha$  large). Thus the linkage principle does not necessarily hold in our set-up, even if the number of bidders is relatively large.

### 3.3 A last example

Our model differs from the standard framework in auction theory (i.e. Milgrom and Weber (1982)) for two reasons: (A) each bidder has a multidimensional signal and (B) the relationship between a bidder's valuation and the seller's signal is not continuous or monotonic. The reader

<sup>9</sup> Indeed  $\partial F/\partial \alpha = \frac{(1-z)\log(1+z)}{(1+\alpha \log(1+z))^2}$  that is strictly positive for  $z \in (0, 1)$ .

may wonder whether (A) or/and (B) are the reasons for which the linkage principle fails in our model. In order to address this issue, we have considered a slightly different framework in which bidders' valuations are given by

$$u(X_i, S) = X_i^1 + SX_i^2,$$

where  $X_i = (X_i^1, X_i^2)$  is the signal observed by bidder  $i$  and  $S$  is the signal observed by the seller. Furthermore we assume that (i)  $X_i^j$  are i.i.d. variables on  $[0, 1]$  with cumulative distribution  $F(\cdot)$  defined by Equation (6) and that (ii)  $S$  is independently distributed from bidders' signals and uniformly distributed on  $[1/2, 1]$ . In this framework all hypotheses of Milgrom and Weber (1982) are satisfied with the exception that bidders receive a bi-dimensional signal. In particular,  $u(\cdot, \cdot)$  is continuous and increasing in  $X_i^1, X_i^2$  and  $S$ .

For any given  $s \in [1/2, 1]$ , the random variable  $U = X_1 + sX_2$  has distribution

$$H(z, s) \stackrel{def}{=} \Pr(U < z) = \int_0^1 F(z - sx_2)f(x_2)dx_2 \quad \forall z \in [0, 1 + s].$$

If the seller discloses  $s$  then bidder  $i$ 's valuation is  $w_i(s) = x_i^1 + sx_i^2$  and the cumulative probability distribution of bidders' valuations is  $H(z, s)$ . Following the same reasoning as in Section 2.1, we deduce that after disclosing his signal  $s$ , the seller's expected revenue is equal to

$$R_D(s) = 1 + s + \int_0^{1+s} (n-1)H(z, s)^n - nH(z, s)^{n-1}dz.$$

Hence the seller's expected revenue (ex-ante) is:

$$R_D = E[R_D(S)] = 1 + E[S] + \int_{1/2}^1 2 \int_0^{1+s} (n-1)H(z, s)^n - nH(z, s)^{n-1}dzds.$$

If the seller does not disclose his signal then bidder  $i$ 's expected valuation is  $w_i = x_i^1 + E[S]x_i^2$ . Hence the cumulative probability distribution of bidders' valuations is  $H(z, E[S])$ . We deduce that the seller's expected revenue is

$$R_{ND} = 1 + E[S] + \int_0^{1+E[S]} (n-1)H(z, E[S])^n - nH(z, E[S])^{n-1}dz = R_D(E[S])$$

In order to understand whether it is optimal for the seller to disclose or not  $S$ , we compute numerically  $R_{ND}$  and  $R_D$ , as we did in Example 2.

$\alpha \backslash n$	2	3	4	5	6	7	8	...	14	15
2	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>
4	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>
10	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>D</i>
30	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>ND</i>	<i>D</i>

**Table 3**

We conclude that the linkage principle does not necessarily apply when bidders have multi-dimensional signals. Notice that as in Examples 1 and 2, the seller is better off not disclosing information when the number of bidders is small and/or the probability of observing  $X_i^j$  close to zero is sufficiently large.

## 4 Conclusion

We conclude by providing the economic intuition for our findings. If the seller reveals the type of the object, then only the bidders who have a high valuation for **this specific** type will aggressively compete for the object. In contrast, if the seller does not reveal the type of the object, then the most aggressive bidders will be those who have high valuation for **some among** the  $m$  possible types of the object. Thus, by keeping secret the type of the object, the seller increases the number of bidders who have large valuations. This effect explains why concealing information can raise the seller's expected revenue. On the other hand, if the seller does not disclose information, a bidder's offer is given by the average of his signals. Hence it is smaller than the highest of his signals. This effect works to decrease the seller's expected revenue and explains why the seller may be better off revealing information.

The first effect dominates the second when, for each object, there is a large probability that the bidders have a small valuation and/or when the number of bidders is small. As this number enlarges, the first effect becomes less important since, for each object, the probability of having at least 2 bidders with large valuations becomes larger. For a sufficiently large number of bidders, the seller is then better off disclosing information.

It is worth stressing that our findings do not depend on the fact that we focus on a second price sealed bid auction. For a given disclosure policy, the revenue-equivalence theorem holds

true in our model (bidders have independent private values for the object). Thus, the ranking of the seller's expected revenue across disclosure regimes extends to the other auction mechanisms that guarantee that the winner is the seller with the highest valuation<sup>10</sup> and that the seller with the lowest possible valuation has an expected payoff equal to zero.

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<sup>10</sup>The ranking of bidders' valuations is done considering the information provided by the seller as fixed. Note that the blind auction is not ex-post efficient as the winner is the bidder with the highest expected valuation, but he is not necessarily the one with the highest ex-post valuation for the object if its type is revealed after the auction.